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INSIGHTS GAINED FROM WEAKLY *P* PROPERTIES INTO EQUIVALENCES OF SEPARATION AXIOMS

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Abstract

In 1975, it was proven that a space is R_1 iff its T_0-identification space is Hausdorff. The 1975 work motivated the introduction and investigation of weakly *Po* properties, which led to the introduction and investigation of weakly *P*1 and weakly *P*2 properties. Within recent papers, it was established that in weakly *P*1 spaces T_0 and T_1 are equivalent, and in weakly *P*2 spaces T_0, T_1, and T_2 are equivalent. Within this paper, infinitely many topological properties, in addition to weakly *P*1 or weakly *P*2, are given for each weakly *P*1 or weakly *P*2 for which each of T_0 and T_1 or T_0, T_1, and T_2 are equivalent, including the least of all such topological property in each of the two cases.

1. Introduction and Preliminaries

T_0-identification spaces were introduced in 1936 [11].

Definition 1.1. Let (X, T) be a space, R be the equivalence relation on X defined by xRy iff $Cl(\{x\}) = Cl(\{y\})$, X_0 be the set of R equivalence

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classes of X, $N : X \to X_0$ be the natural map, and $Q(X, T)$ be the decomposition topology on X_0 determined by (X, T) and the natural map N. Then $(X_0, Q(X, T))$ is the T_0-identification space of (X, T) [11].

Within the 1975 paper [10], weakly Hausdorff was characterized using T_0-identification spaces.

Theorem 1.1. *A space (X, T) is weakly Hausdorff iff its T_0-identification space is Hausdorff* [10].

In the 2015 paper [2], the question of whether T_0-identification spaces could be used to uniquely define other weakly P properties behaving in the same manner as weakly Hausdorff led to the introduction and investigation of weakly Po properties.

Definition 1.2. Let P be a topological property for which $Po = (P$ and $T_0)$ exists. Then (X, T) is weakly Po iff its T_0-identification space $(X_0, Q(X, T))$ has property P. A topological property Po for which weakly Po exists is called a weakly Po property [2].

In the 2015 paper [2], it was proven that for a topological property P for which weakly Po exists, weakly Po is a unique, topological property. In addition, since for each space (X, T), $(X_0, Q(X, T))$ is T_0 [11], then, as given in the 2015 paper [2], a space is weakly Po iff its T_0-identification space has property Po.

Within the 1975 paper [10], it was proven that weakly Hausdorff is equivalent to the R_1 separation axiom, which was introduced in 1961 [1].

Definition 1.3. A space (X, T) is R_1 iff for $x, y \in X$, such that $Cl(\{x\}) \neq Cl(\{x\})$, there exist disjoint open sets U and V such that $x \in U$ and $y \in V$ [1].

In the 1961 paper [1], the following characterizations of T_2 were given: For a space (X, T), the following are equivalent: (a) (X, T) is T_2, (b) (X, T) is $(R_1$ and $T_1)$, and (c) (X, T) is $(R_1$ and $T_0)$.

Also, within the 1961 paper [1], the R_0 separation axiom was rediscovered and used to further characterize T_1 spaces.

Definition 1.4. A space (X, T) is R_0 iff for each open set *O* and each $x \in O$, $Cl(\{x\}) \subseteq O$.

In the 1961 paper [1], it was shown that R_1 implies R_0 and a space is T_1 iff it is (R_0 and T_0).

Within weakly *Po* properties, the T_0 separation axiom has a major role. Thus the question of what would happen if T_0 in the definition of weakly *Po* properties was replaced by T_1 or by T_2 arose leading to the introduction and investigation of weakly *P*1 and weakly *P*2 properties.

Definition 1.5. Let *P* be a topological property for which $P1 = (P$ and $T_1)$ exists. Then a space (X, T) is weakly *P*1 iff its T_0-identification space $(X_0, Q(X, T))$ has property *P*1 [3].

Definition 1.6. Let *P* be a topological property for which $P2 = (P$ and $T_2)$ exists. Then a space (X, T) is weakly *P*2 iff its T_0-identification space $(X_0, Q(X, T))$ has property *P*2 [4].

The continued investigation of weakly *P*1 properties [5] has revealed that for weakly *P*1 spaces, T_0 and T_1 are equivalent and for weakly *P*2 spaces, T_0, T_1, and T_2 are all equivalent [6]. Thus, the question of whether there are additional topological properties for which T_0 and T_1 are equivalent and for which T_0, T_1, and T_2 are all equivalent arises. The equivalences given above were not expected, but, in this case, as is often true, discoveries lead to additional questions for consideration and hopefully resolution. Below for each weakly *P*1 property and for each weakly *P*2 property, infinitely many related topological properties are given in which the equivalences above hold.

2. Related Topological Properties for a Weakly *P*1 and Weakly *P*2 Property Preserving the Equivalences

Confronted by a new problem, mathematicians are trained to search for a way to relate the new problem to an already solved problem and, if successful, use the solution of the solved problem to aid in the solution of the new problem. For the cited new problems above, there are solved problems that can be applied to greatly aid in their resolution.

Within the 2016 paper [7], the following result was proven: "Let Q be a topological property for which weakly Qo exists and let $\mathcal{S} = \{So \mid S$ is a topological property, So exists, and So implies $Qo\}$. Then (weakly $Qo)o \in \mathcal{S}$, for each weakly Po property W such that Wo implies Qo, (weakly $Wo)o \in \mathcal{S}$, each element of $\mathcal{S}$ implies weakly Qo, and there exists the topological property $Q_{\min}$ = ((weakly Qo) or "not-T_0"), where "not-T_0" is the negation of T_0, weaker than weakly Qo such that $(Q_{\min})o \in \mathcal{S}$". In the weakly $P1$ paper [3], it was shown that for a weakly $P1$ property $Q1$, $Q1$ is a weakly Po property, i.e., $Q1 = (Q1)o$, which is combined with the results above to quickly establish the following useful results in the resolution of the new problems above.

Corollary 2.1. *Let Q be a topological property for which weakly $Q1$ exists and let $\mathcal{S}(Q1) = \{So \mid S$ is a topological property, So exists, and So implies $Q1\}$. Then $\mathcal{S}(Q1) = \{So \mid S$ is a topological property, So exists, and So implies $(Q1)o\}$.*

Corollary 2.2. *Let Q be a topological property for which weakly $Q1$ exists and let $\mathcal{S}(Q1) = \{So \mid S$ is a topological property, So exists, and So implies $Q1\}$. Then (weakly $Q1)o \in \mathcal{S}(Q1)$, for each weakly Po property W such that Wo implies $Q1$, (weakly $Wo)o \in \mathcal{S}(Q1)$, each element of $\mathcal{S}(Q1)$ implies weakly $Q1$, and there exists the topological property $(Q_{\min})1$ = ((weakly $Q1$) or "not-T_0") weaker than weakly $Q1$ such that $((Q_{\min})1)o \in \mathcal{S}(Q1)$.*

Also, in the 2016 paper [7], it was established that for a topological property for which weakly *Po* exists, $P_{\min}$ is the least topological property for which a space has property *Po* iff it has property ($P_{\min}$ and T_0), giving the next result.

Corollary 2.3. *Let P be a topological property for which weakly P1 exists. Then* $(P_{\min})1$ *is the least topological property for which a space has property P1 iff it has properties* $((P_{\min})1$ *and* T_0).

Within the second 2016 paper [8], for a weakly *Po* property *Qo*, the special role played by $Q_{\min}$ was used to give infinitely many topologically distinct, non-weakly *Po* topological properties weaker than weakly *Qo* and stronger than $Q_{\min}$, which together with T_0, are equivalent to *Qo*, which will be combined with the results above to extend the 2016 result [8] to weakly *P*1 properties.

Let *m* and *n* represent natural number greater than or equal to 2.

Definition 2.1. Let $A(n)$ represent a set with *n* distinct elements, *X* be a set containing the elements of $A(n)$, and $T(A(n))$ be the topology on *X* defined by $T(A(n)) = \{B \subseteq X \mid A(n) \subseteq B \text{ or } B = \phi\}$ [8].

Definition 2.2. A space (X, T) has property $T(n)$ iff there exists a subset $A(n)$ of *X* such that $T = T(A(n))$ [8].

In the 2016 paper [8], it was shown that each $T(n)$ space is "not-T_0" and not a weakly *Po* property, that $Q(n) = (\text{weakly } Qo \text{ or } T(n))$ is a topological property weaker than weakly *Qo* and stronger than $Q_{\min}$ such that a space has property *Qo* iff it has property ($Q(n)$ and T_0), and that for $m < n$, $Q(m)$ and $Q(n)$ are distinct topological properties, which is combined with the results above to give the next result.

Corollary 2.4. *For each n and each weakly $P1$ property $Q1$, $Q(n)1$ = ((weakly $Q1$) or $T(n)$) is a non-weakly $P1$ topological property weaker than weakly $Q1$ and stronger than $(Q_{\min})1$ such that a space has property $Q1$ iff it has property $((Q(n)1)$ and $T_0)$.*

Thus, there are infinitely many non-weakly $P1$ topological properties W weaker than weakly $Q1$ and stronger than $(Q_{\min})1$ such that a space has property $Q1$ iff it has property $(W$ and $T_0)$.

In the first 2016 paper [7], it was shown that for a weakly Po property Qo, $Q_{(\min, \max)}$ = ((weakly Qo) and "not-T_0") also plays a special role: $Q_{(\min, \max)}$ is the least topological property weaker than Qo and stronger than weakly Qo such that a space is Qo iff it is $(Q_{(\min, \max)}$ and $T_0)$. Combining this result with those above give the following result.

Corollary 2.5. *Let Q be a topological property for which weakly $Q1$ exists. Then $(Q_{(\min, \max)})1$ = ((weakly $Q1$) and "not-T_0") is the least topological property weaker than $Q1$ and stronger than weakly $Q1$ such that a space is $Q1$ iff it is $((Q_{(\min, \max)})1$ and $T_0)$.*

Within the second 2016 paper [8], for each weakly Po property Qo, $Q(1, n)$ was defined and used to give infinitely many more topological properties, which together with T_0, are equivalent to Qo.

Definition 2.3. Let Q be a topological property for which weakly Qo exists. A space (X, T) is $Q(1, n)$ iff it is weakly Qo, there exist n distinct elements $a_1, \cdots, a_n$ all of whose closures are equal, and for all other $x \in X$, $Cl(\{x\}) = Cl(\{y\})$ iff $x = y$ [8].

In the 2016 paper [8], for a weakly Po property Qo, it was shown that $Q(1, n)$ exists, $Q(1, n)$ is weaker than Qo and stronger than $Q_{(\min, \max)}$, $Qo = (Q(1, n)$ and $T_0)$, and for each Qo space (Y, S) there are infinitely many topologically spaces all with topologically distinct topological properties that are weaker than Qo and stronger than $Q_{(\min, \max)}$, which together with T_0, equals Qo and all having a T_0-identification space homeomorphic to (Y, S).

Definition 2.4. Let Q be a topological property for which weakly $Q1$ exists. A space (X, T) is $(Q(1, n))1$ iff it is weakly $Q1$, there exist n distinct elements $a_1, \cdots, a_n$ all of whose closures are equal, and for all other $x \in X$, $Cl(\{x\}) = Cl(\{y\})$ iff $x = y$.

Corollary 2.6. *Let Q be a topological property for which weakly $Q1$ exists. Then $(Q(1, n))1$ exists, $(Q(1, n))1$ is weaker than $Q1$ and stronger than $(Q_{(\min, \max)})1$, $Q1 = ((Q(1, n))1$ and $T_0)$, and for each $Q1$ space (Y, S), there are infinitely many topologically spaces all with topologically distinct topological properties that are weaker than $Q1$ and stronger than $(Q_{(\min, \max)})1$, which together with T_0, equals $Q1$ and all having a T_0-identification space homeomorphic to (Y, S).*

As indicated in the paper [8], if (Y, S) has property Qo, where Qo is a weakly Po property, with p or more elements, then $Q(1, n)$ can be extended to $Q(p, n_1, \cdots, n_p)$ that behaves in the same manner as $Q(1, n)$ and can be used to give many more topological properties weaker than Qo and stronger than $Q_{(\min, \max)}$, which together with T_0, is equivalent to Qo. In the same manner as above, each of these new topological properties can be used to give a topological property weaker than $Q1$ and stronger than $(Q_{(\min, \max)})1$, which together with T_0, is equivalent to $Q1$.

Theorem 2.1. *Let P be a topological property such that $(P$ and $T_0)$ exists. Then $(P$ and $T_0)$ implies $P1$ iff T_0 and T_1 are equivalent.*

Proof. Suppose (P and T_0) implies $P1$. Thus, if (X, T) is a P space with property T_0, then (X, T) has property $P1$, which implies (X, T) is T_1. Since T_1 implies T_0, then T_0 and T_1 are equivalent.

Clearly, the converse is true.

Corollary 2.7. *Let Q be a topological property for which weakly $Q1$ exists. Then for each topological property P given above for which (P and T_0) = $Q1$, T_0 and T_1 are equivalent.*

Since for a topological property Q for which weakly $Q2$ exists, $Q2 = (Q2)o$ [6], then each of the results above can be correctly restated by replacing weakly $Q1$ by weakly $Q2$ and $Q1$ by $Q2$ and since for a topological property P for which (P and T_0) exists, (P and T_0) implies $P2$ iff T_0, T_1, and T_2 are equivalent, then there are infinitely many known topologically distinct topological properties for which each of T_0, T_1, and T_2 are equivalent for each weakly $P2$ property.

Of the many topological properties P for which (P and T_0) exists and T_0 and T_1 are equivalent, is there a least such topological property and of the many topological properties P for which (P and T_0) exists and T_0, T_1, and T_2 are equivalent, is there a least such topological property? These questions are resolved in the last section of this paper.

3. The Least Topological Properties Preserving the Equivalences

Theorem 3.1. *The least of all topological properties P for which (P and T_0) exists and T_0 and T_1 are equivalent is (R_0 or "not-T_0").*

Proof. Let $\mathcal{P}1 = \{So \mid S$ is a topological property, So exists, and So implies $T_1\}$. By the results above, ((weakly T_1) or "not-T_0") is the least topological property P for which a space has property T_1 iff it is (P and T_0). Since weakly T_1 = weakly $(R_0)o$ [2], then P = ((R_0) or "not-T_0") is the least of all topological properties P for which a space is T_1 iff it is (P and T_0). Thus ((R_0) or "not-T_0") is the least topological property P for which (P and T_0) exists and T_0 and T_1 are equivalent.

Theorem 3.2. *The least of all topological properties P for which (P and T_0) exists and T_0, T_1, and T_2 are equivalent is ((R_1) or "not-T_0").*

Proof. Let $\mathcal{P}2 = \{So|\ S$ is a topological property, So exists, and So implies $T_2\}$. By the results above, ((weakly T_2) or "not-T_0") is the least topological property P for which a space has property T_2 iff it is (P and T_0). Since weakly T_2 = weakly $(R_1)o$ [2], then P = ((R_1) or "not-T_0") is the least of all topological properties P for which a space is T_2 iff it is (P and T_0). Thus ((R_1) or "not-T_0") is the least topological property P for which (P and T_0) exists and T_0 and T_1 are equivalent.

The last two results follow immediately from the results above.

Corollary 3.1. *Let (X, T) be R_0. Then (X, T) is T_0 iff (X, T) is T_1.*

Corollary 3.2. *Let (X, T) be R_1. Then (X, T) is T_0 iff (X, T) is T_1 iff (X, T) is T_2.*

Thus, knowledge of solved problems that are related to the problems above and then applying that knowledge has made difficult problems quickly solvable. In the 1961 paper [1], the focus was the separation axioms T_1 and T_2. The introduction and investigation of weakly *Po*, weakly *P*1, and weakly *P*2 properties has changed the focus from T_1 and T_2 to R_0, R_1, and infinitely many new topological properties that add much and give greater insight into the working of mathematics. With the focus on T_1 and T_2 in 1961 [1], one characterization of T_1 was given and two characterizations of T_2 was given and, even though the knowledge was available, use of R_0 and R_1 in the equivalence questions above were overlooked. However, as a result of the 1961 paper [1] and the 1975 paper [10], today infinitely many topological properties are known for which the equivalences above are true and infinitely many new characterizations of T_1 and T_2 have been discovered [9].

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