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SOME OPERATIONS ON $\mathbb{B}^{-1}$-CONVEX SETS

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Abstract

$\mathbb{B}^{-1}$-convexity is an abstract convexity type. $\mathbb{B}^{-1}$-convex sets are examined in various studies. Also, the applications of $\mathbb{B}^{-1}$-convexity on mathematical economy are introduced in some new papers. In this article, some operations on $\mathbb{B}^{-1}$-convex sets are proved.

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1. Introduction

Recently, abstract convexity which has many applications to the mathematical economy, operation research, inequality theory is a popular area in mathematics ([5, 6, 7]). Besides, $\mathbb{B}^{-1}$-convexity is an abstract convexity type. $\mathbb{B}^{-1}$-convex sets are examined in various studies ([1, 2, 3, 4, 8]). Also, the applications of $\mathbb{B}^{-1}$-convexity on mathematical economy are introduced in [3]. In this article, some operations on $\mathbb{B}^{-1}$-convex sets are proved.

In Section 2, we give definitions and recall some properties of $\mathbb{B}^{-1}$-convex set and $\mathbb{B}^{-1}$-convex hull. In Section 3, we establish some new operations on $\mathbb{B}^{-1}$-convex sets.

2. $\mathbb{B}^{-1}$-Convexity and Operations on $\mathbb{B}^{-1}$-Convex Sets

For $r \in \mathbb{Z}^{-}$, the map $x \to \varphi_r(x) = x^{2r+1}$ is a homeomorphism from $K = \mathbb{R} \setminus \{0\}$ to itself; $\boldsymbol{x} = (x_1, x_2, \ldots, x_n) \to \Phi_r(\boldsymbol{x}) = (\varphi_r(x_1), \varphi_r(x_2), \ldots, \varphi_r(x_n))$ is homeomorphism from K^n to itself.

For a finite nonempty set $A = \left\{\boldsymbol{x}^{(1)}, \boldsymbol{x}^{(2)}, \ldots, \boldsymbol{x}^{(\boldsymbol{m})}\right\} \subset K^n$ the Φ_r-convex hull (shortly r-convex hull) of A, which we denote $Co^r(A)$ is given by

$$Co^r(A) = \left\{\Phi_r^{-1}\left(\sum_{i=1}^{m} t_i \Phi_r(\boldsymbol{x}^{(\boldsymbol{i})})\right) : t_i \geq 0, \sum_{i=1}^{m} t_i = 1\right\}.$$

Thus, we can define $\mathbb{B}^{-1}$-polytopes as follows:

Definition 2.1 ([1]). The Kuratowski-Painleve upper limit of the sequence of sets $\{Co^r(A)\}_{r \in \mathbb{Z}^-}$, denoted by $Co^{-\infty}(A)$, where A is a finite subset of K^n, is called $\mathbb{B}^{-1}$-polytope of A.

Next, we give the definition of $\mathbb{B}^{-1}$-convex sets.

Definition 2.2 ([1])**.** A subset U of K^n is called a $\mathbb{B}^{-1}$-convex if for all finite subsets $A \subset U$ the $\mathbb{B}^{-1}$-polytope $Co^{-\infty}(A)$ is contained in U.

We denote by $\overset{m}{\underset{i=1}{\wedge}} \boldsymbol{x}^{(\boldsymbol{i})}$ the greatest lower bound with respect to the coordinate-wise order relation of $\boldsymbol{x}^{(1)}, \boldsymbol{x}^{(2)}, \ldots, \boldsymbol{x}^{(\boldsymbol{m})} \in \mathbb{R}^n$, that is;

$$\overset{m}{\underset{i=1}{\wedge}} \boldsymbol{x}^{(i)} = \left(\min\left\{x_1^{(1)}, x_1^{(2)}, \ldots, x_1^{(m)}\right\}, \ldots, \min\left\{x_n^{(1)}, x_n^{(2)}, \ldots, x_n^{(m)}\right\}\right),$$

where $x_j^{(i)}$ denotes j-th coordinate of the point $\boldsymbol{x}^{(\boldsymbol{i})}$.

The definition of $\mathbb{B}^{-1}$-polytope can be expressed in the following form in $\mathbb{R}^n_{++} = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_i > 0, i = 1, 2, \ldots, n\}$.

Theorem 2.1 ([1])**.** *For all nonempty finite subsets* $A = \{\boldsymbol{x}^{(1)}, \boldsymbol{x}^{(2)}, \ldots, \boldsymbol{x}^{(\boldsymbol{m})}\} \subset \mathbb{R}^n_{++}$, *we have*

$$Co^{-\infty}(A) = \lim_{r \to -\infty} Co^r(A) = \left\{\overset{m}{\underset{i=1}{\wedge}} t_i \boldsymbol{x}^{(\boldsymbol{i})} : t_i \geq 1, \min_{1 \leq i \leq m} t_i = 1\right\}.$$

By Theorem 2.1, we can reformulate the above definition for subsets of $\mathbb{R}^n_{++}$.

Theorem 2.2 ([1])**.** *A subset U of* $\mathbb{R}^n_{++}$ *is* $\mathbb{B}^{-1}$*-convex if and only if for all* $\boldsymbol{x}^{(1)}, \boldsymbol{x}^{(2)} \in U$ *and all* $\lambda \in [1, \infty)$ *one has* $\lambda \boldsymbol{x}^{(1)} \wedge \boldsymbol{x}^{(2)} \in U$.

The following properties of $\mathbb{B}^{-1}$-convexity are given in [2]:

Theorem 2.3. (a) *The empty set,* K^n, *as well as the singletons are* $\mathbb{B}^{-1}$*-convex*;

(b) *if* $\{S_\lambda : \lambda \in \Lambda\}$ *is an arbitrary family of* $\mathbb{B}^{-1}$*-convex sets, then* $\bigcap_\lambda S_\lambda$ *is* $\mathbb{B}^{-1}$*-convex*;

(c) *if* $\{S_\lambda : \lambda \in \Lambda\}$ *is a family of* $\mathbb{B}^{-1}$*-convex sets such that* $\forall \lambda_1, \lambda_2 \in \Lambda$, $\exists \lambda_3 \in \Lambda$ *such that* $S_{\lambda_1} \bigcup S_{\lambda_2} \subset S_{\lambda_3}$, *then* $\bigcup_\lambda S_\lambda$ *is* $\mathbb{B}^{-1}$*-convex.*

Given a set $S \subset K^n$, the intersection of all the $\mathbb{B}^{-1}$-convex subsets of K^n containing S is called the $\mathbb{B}^{-1}$-convex hull of S and is denoted by $\mathbb{B}^{-1}[S]$.

The theorem related to properties of $\mathbb{B}^{-1}$-convex hull is given in [2].

Theorem 2.4. *The following properties hold*:

(a) $\mathbb{B}^{-1}[\emptyset] = \emptyset$, $\mathbb{B}^{-1}[K^n] = K^n$, *for all* $x \in K^n$, $\mathbb{B}^{-1}[\{x\}] = \{x\}$;

(b) *For all* $S \subset K^n$, $S \subset \mathbb{B}^{-1}[S]$ *and* $\mathbb{B}^{-1}[\mathbb{B}^{-1}[S]] = \mathbb{B}^{-1}[S]$;

(c) *For all* $S_1, S_2 \subset K^n$, *if* $S_1 \subset S_2$, *then* $\mathbb{B}^{-1}[S_1] \subset \mathbb{B}^{-1}[S_2]$;

(d) *For all* $S \subset K^n$, $\mathbb{B}^{-1}[S] = \bigcup\{\mathbb{B}^{-1}[A] :$ A *is a finite subset of* $S\}$;

(e) *A subset* $S \subset K^n$ *is* $\mathbb{B}^{-1}$*-convex if and only if for all finite subset* A *of* S, $\mathbb{B}^{-1}[A] \subset S$.

3. Other Operations on $\mathbb{B}^{-1}$-Convex Sets

It can be seen that in $\mathbb{R}^n_{++}$ for a finite set A, $Co^{-\infty}(A)$ is a $\mathbb{B}^{-1}$-convex set from the following theorem:

Theorem 3.1. *If A is a finite subset of $\mathbb{R}^n_{++}$, then $Co^{-\infty}(A)$ is $\mathbb{B}^{-1}$-convex.*

Proof. Let $A = \{\boldsymbol{x}^{(1)}, \boldsymbol{x}^{(2)}, \dots, \boldsymbol{x}^{(m)}\} \subset \mathbb{R}^n_{++}$, $\boldsymbol{x} = \wedge_{j=1}^m \rho_j \boldsymbol{x}^{(j)}$, $\boldsymbol{y} = \wedge_{j=1}^m \eta_j \boldsymbol{x}^{(j)}$ with $(\rho_1, \rho_2, \dots, \rho_m), (\eta_1, \eta_2, \dots, \eta_m) \in [1, +\infty)^m$ and $\min\{\rho_1, \rho_2, \dots, \rho_m\} = \min\{\eta_1, \eta_2, \dots, \eta_m\} = 1$; both $\boldsymbol{x}$ and $\boldsymbol{y}$ are two elements of $Co^{-\infty}(A)$.

We have to see that $Co^{-\infty}(\{\boldsymbol{x}, \boldsymbol{y}\}) \subset Co^{-\infty}(A)$. Let $u \in Co^{-\infty}(\{\boldsymbol{x}, \boldsymbol{y}\})$; there exists $(\mu_1, \mu_2) \in [1, +\infty)^2$ with $\min\{\mu_1, \mu_2\} = 1$ such that $u = \mu_1 \boldsymbol{x} \wedge \mu_2 \boldsymbol{y}$.

$$u = \mu_1\left(\bigwedge_{j=1}^{m} \rho_j \boldsymbol{x}^{(j)}\right) \wedge \mu_2 (\wedge_{j=1}^m \eta_j \boldsymbol{x}^{(j)}) = \bigwedge_{j=1}^{m} \min\{\mu_1\rho_j, \mu_2\eta_j\} \boldsymbol{x}^{(j)}.$$

To conclude the proof, just notice that $\min_{1 \le j \le m}\{\min\{\mu_1\rho_j, \mu_2\eta_j\}\} = 1$.

□

Remark 3.1. For an arbitrary finite set $A \subset \mathbb{R}^n_{++}$, its $\mathbb{B}^{-1}$-convex hull $\mathbb{B}^{-1}[A]$ is a $\mathbb{B}^{-1}$-convex set and $A \subset \mathbb{B}^{-1}[A]$. From the definition of $\mathbb{B}^{-1}$-convex set and $A \subset \mathbb{B}^{-1}[A]$, we have $Co^{-\infty}(A) \subset \mathbb{B}^{-1}[A]$.

Also the set $Co^{-\infty}(A)$ is $\mathbb{B}^{-1}$-convex and $A \subset Co^{-\infty}(A)$.

Since $\mathbb{B}^{-1}$-convex hull of A is the smallest $\mathbb{B}^{-1}$-convex set containing A, we obtain that $\mathbb{B}^{-1}[A] = Co^{-\infty}(A)$.

Theorem 3.2. *Let $L \subset \mathbb{R}^n_{++}$ and denote by $\langle L \rangle$ be the family of nonempty finite subsets of L, then*

$$\mathbb{B}^{-1}[L] = \bigcup_{A \in \langle L \rangle} Co^{-\infty}(A).$$

Proof. Clearly, from (d) of Theorem 2.4, we have $\mathbb{B}^{-1}[L] = \bigcup\{\mathbb{B}^{-1}[A] : A \in \langle L\rangle\}$ and we have shown above that $\mathbb{B}^{-1}[A] = Co^{-\infty}(A)$ for $A \subset \mathbb{R}^n_{++}$. $\square$

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