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APPLICATION OF WEAKLY *P* PROPERTIES GIVING INFINITELY MANY NEW CHARACTERIZATIONS OF T_1 AND T_2 SPACES

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Abstract

Within this paper, the important separation axioms of T_0, T_1, and T_2 are further investigated with infinitely many new characterizations of T_1 and T_2 given using T_0, R_0, R_1, and known weakly P properties.

1. Introduction and Preliminaries

In the study of topology each of T_0, T_1, and T_2 have an important role. The T_0 separation axiom is precisely the topological property separating metrizable spaces and pseudometrizable spaces, the T_1 separation axiom is precisely the topological property requiring singleton sets to be closed, and the T_2 separation axiom is the

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topological property guaranteeing uniqueness of convergence. As a result of the importance of the T_i separation axioms; $i = 0, 1, 2,$ in 1961 [1], Davis was interested in topological properties R_i weaker than T_{i+1}, which together with T_i, is equivalent to T_{i+1}; $i = 0, 1$.

The R_0 axiom was introduced in 1943 [6] and rediscovered by Davis in 1961.

Definition 1.1. A space (X, T) is R_0 iff for each closed set C and each $x \notin C$, $C \cap Cl(\{x\}) = \phi$.

Definition 1.2. A space (X, T) is R_1 iff for $x, y \in X$ such that $Cl(\{x\}) \neq Cl(\{y\})$, there exist disjoint open sets U and V such that $x \in U$ and $y \in V$ [1].

Theorem 1.1. *A space is* T_{i+1} *iff it is* $(R_i$ *and* $T_i)$, $i = 0, 1$, *respectively* [1].

In 1975 [5], Dunham used T_0-identification spaces to once again give a direct connection between T_2 and R_1.

T_0-identification spaces were introduced in 1936 [7].

Definition 1.3. Let (X, T) be a space, let R be the equivalence relation on X defined by xRy iff $Cl(\{x\}) = Cl(\{y\})$, let X_0 be the set of R equivalence classes of X, let $N : X \to X_0$ be the natural map, and let $Q(X, T)$ be the decomposition topology on X_0 determined by (X, T) and the natural map N. Then $(X_0, Q(X, T))$ is the T_0-identification space of (X, T) [7].

Theorem 1.2. *A space* (X, T) *is weakly Hausdorff iff its* T_0*-identification space is Hausdorff* [5].

Within the 1975 paper [5], it was shown that weakly Hausdorff and R_1 are equivalent separation axioms.

In the 1936 paper [7], it was shown that a space is pseudometrizable iff its T_0-identification space is metrizable. Thus the question: "Could the process used to further characterize pseudometrizable and R_1 using T_0-identification space be generalized to further characterize other topological properties?" arose leading to the introduction and investigation of weakly P properties [2].

Definition 1.4. A topological property P is a weakly P property iff a space (X, T) has property P iff its T_0-identification space $(X_0, Q(X, T))$ has property P [2].

Since for each space (X, T) its T_0-identification space is T_0 [7], then for a topological property P to even be considered for a weakly P property, (P and T_0) must exist. In this paper, for a topological property P for which (P and T_0) exist, Po is conveniently used in place of (P and T_0).

Thus, for a weakly P property Q, a space has property Q iff its T_0-identification space has property Qo = (Q and T_0), allowing weakly P and weakly Po to be used interchangeably.

Since (pseudometrizable and T_0) = metrizable, then, long before the definition of weakly P properties, pseudometrizable was the first known weakly P property with pseudometrizable = weakly (pseudometrizable) = weakly (pseudometrizable)o = weakly (metrizable) and R_1 was the second known weakly P property with R_1 = weakly R_1 = weakly (R_1)o = weakly T_2. Within the initial weakly P paper [2], it was shown that R_0 is another weakly P property with R_0 = weakly R_0 = weakly (R_0)o = weakly T_1.

Below, properties of weakly P properties are applied giving infinitely many new characterizations of the T_1 and T_2 separation axioms showing R_i in Theorem 1.1 is not unique; $i = 0, 1,$ respectively.

2. Infinitely Many New Characterizations of T_1 Spaces

Within a 2016 paper [3], the following results were proven.

Theorem 2.1. *Let Q be a topological property for which weakly Qo exists and let $\mathcal{S} = \{S\text{o} \mid S$ is a topological property, So exists, and So implies $Q\text{o}\}$. Then $(\text{weakly } Q\text{o})\text{o} \in \mathcal{S}$, for each weakly Po property W such that Wo implies Qo, $(\text{weakly } W\text{o})\text{o} \in \mathcal{S}$, each element of $\mathcal{S}$ implies weakly Qo, and there exists a topological property $Q_{min} = ((\text{weakly } Q\text{o})$ or "not-T_0") weaker than weakly Qo such that $(Q_{min})\text{o} \in \mathcal{S}$, where "not-$T_0$" is the negation of T_0* [3].

Theorem 2.2. *Let Q and $\mathcal{S}$ be as in Theorem* 1.2. *Then Q_{min} = $((\text{weakly } Q\text{o})$ or "not-T_0") is the least topological property P weaker than weakly Qo such that $P\text{o} \in \mathcal{S}$* [3]. *Clearly, by the definition of Q_{min} given above, Q_{min} is weaker than weakly Qo and thus Q_{min} is the least element of $\mathcal{S}$.*

Applying these results to the question above gives the following answer.

Replacing Q in the Theorems above by R_0 gives the next two results.

Corollary 2.1. *Let $\mathcal{S} = \{S\text{o} \mid S$ is a topological property, So exists, and So implies $T_1\}$. Then $(R_0)\text{o} = T_1 \in \mathcal{S}$, for each weakly Po property W such that Wo implies T_1, $(\text{weakly } W\text{o})\text{o} \in \mathcal{S}$, each element of $\mathcal{S}$ implies T_1, $(R_0)_{min} = (R_0$ or "not-T_0") is weaker than R_0, $(R_0)_{min}$ is the least element of $\mathcal{S}$, and $(((R_0)_{min})$ and $T_0) = T_1$.*

Theorem 2.3. *$(R_0)_{min}$ is the least topological property W such that $(W$ and $T_0) = T_1$ and, thus R_0 is not unique in the statement of Theorem* 1.1.

Proof. Let W be a topological property such that $(W \text{ and } T_0) = T_1$. Then $W\mathrm{o} \in \mathcal{S}$, as given in Corollary 2.1, and $(R_0)_{\min}$ is weaker than or equal to W.

Motivated by Theorems 2.1 and 2.2 above, within a follow up 2016 paper [4] attention was then turned to the question: "Are there topological properties Q between weakly $P\mathrm{o}$ and $P_{\min}$ for which a space has property $P\mathrm{o}$ iff it has property $(Q \text{ and } T_0)$?", which led to the definitions and results below.

Below m and n are used to represent natural numbers greater than or equal to 2.

Definition 2.1. Let $A(n)$ represent a set with n distinct elements, let X be a set containing $A(n)$, and let $T(A(n))$ be the topology on X defined by $T(A(n)) = \{B \subseteq X \mid A(n) \subseteq B \text{ or } B = \phi\}$ [4].

Definition 2.2. A space (X, T) has property $T(n)$ iff there exists a subset $A(n)$ of X such that $T = T(A(n))$ [4].

Theorem 2.4. *$T(n)$ is a topological property, for each $T(n)$ space (X, T), (X, T) is "not-T_0", and $T(n)$ is not a weakly $P\mathrm{o}$ property* [4].

Theorem 2.5. *Let m be less than n. Then $T(m)$ and $T(n)$ are topologically distinct topological properties* [4].

Theorem 2.6. *Let Q be a topological property for which weakly $Q\mathrm{o}$ exists. Then $Q(n) = (\text{weakly } Q\mathrm{o} \text{ or } T(n))$ is a topological property weaker than weakly $Q\mathrm{o}$ and stronger than Q_{min} such that a space has property $Q\mathrm{o}$ iff it has property $(Q(n) \text{ and } T_0)$* [4].

Theorem 2.7. *Let $m < n$. Then $Q(m)$ and $Q(n)$ are distinct topological properties* [4]. *Replacing Q in Theorem 2.6 by R_0 gives the following new characterization of T_1 spaces.*

Corollary 2.2. *$(R_0)(n) = ((R_0)$ or $T(n))$ is a topological property weaker than R_0 and stronger than $(R_0)_{min}$ such that a space has property T_1 iff it has property $((R_0)(n)$ and $T_0)$.*

Combining Theorem 2.7 with Corollary 2.2 gives infinitely many new characterizations of T_1 spaces.

Within the paper [3], the following result was proven:

Theorem 2.8. *Let Q be a weakly P property. Then $Q_{(min, max)} = ((weakly\ Q)$ and "not-T_0") is the least topological property weaker than Qo and stronger than weakly Q such that a space is Qo iff it is $(Q_{(min, max)}$ and $T_0)$* [3].

Replacing Q in Theorem 2.8 by R_0 gives the following new characterization of T_1.

Corollary 2.3. *$(R_0)_{(min,max)} = (R_0$ and "not-T_0") is the least topological property weaker than T_1 and stronger than R_0 such that a space is T_1 iff it is $((R_0)_{(min, max)}$ and $T_0)$.*

After Theorem 2.8 in the paper [4], the following question was raised: "Are there topological properties W between Qo and $Q_{(\min, \max)}$ for which a space is Qo iff it has property $(W$ and $T_0)$?" leading to the definitions and results below.

Definition 2.3. Let Q be a topological property for which weakly Q exists. A space (X, T) is $Q(1, n)$ iff it is weakly Q, there exist n distinct elements $a_1, \cdots, a_n$ all of whose closures are equal, and for all other $x \in X$, $Cl(\{x\}) = Cl(\{y\})$ iff $x = y$ [4].

Theorem 2.9. *Let Q be a topological property for which weakly Qo exists. Then $Q(1, n)$ exists and is a topological property* [4].

Theorem 2.10. *Let* $m < n$. *Then* $Q(1, m)$ *and* $Q(1, n)$ *are topologically distinct topological properties* [4].

Theorem 2.11. *Let* Q *be a topological property for which weakly* $Q\text{o}$ *exists. Then* $Q(1, n)$ *is weaker than* $Q\text{o}$ *and stronger than* $Q_{(min, max)}$, *and* $Q\text{o} = (Q(1, n)\, and\ T_0)$ [4].

Theorem 2.12. *Let* Q *be a topological property for which weakly* $Q\text{o}$ *exists. Then for each* $Q\text{o}$ *space* (Y, S), *there are infinitely many topologically spaces all with topologically distinct topological properties that are weaker than* $Q\text{o}$ *and stronger than* $Q_{(min, max)}$, *which together with* T_0, *equals* $Q\text{o}$ *and all having a* T_0*-identification space homeomorphic to* (Y, S) [4].

Replacing Q in Theorems 2.11 and 2.12 by R_0 gives more new characterizations of T_1.

Corollary 2.4. $(R_0)(1, n)$ *is weaker than* T_1 *and stronger than* $(R_0)_{(min, max)}$ *and* $T_1 = ((R_0)(1, n)$ *and* $T_0)$.

Corollary 2.5. *For each* T_1 *space* (Y, S), *there are infinitely many topologically spaces all with topologically distinct topological properties that are weaker than* T_1 *and stronger than* $(R_0)_{(min, max)}$, *which together with* T_0, *equals* T_1 *and all having a* T_0*-identification space homeomorphic to* (Y, S).

Lastly within the paper [4], the question of whether, in certain case, $Q(1, n)$ for a weakly P property Q could be extended to obtain even more characterization of $Q\text{o}$ was addressed as given below.

If (Y, S) is a $Q\text{o}$ space, where Q is a topological property for which weakly Q exists, having two or more elements, then to a selected elements a_1 in Y elements $a_2, \cdots, a_m$ could be added and to another selected element b_1 in Y elements $b_2, \cdots, b_n$ could be added, $m \leq n$, and used to define $Q(2, m, n)$ that would behave in the same manner as $Q(1, n)$. If (Y, S) has three or more elements, $Q(3, m, n, p), m \leq n \leq p$, could be defined that would behave the same manner as $Q(1, n)$. This process could be continued for Y sufficiently large giving many more topologically distinct topological properties all behaving in the same manner as $Q(1, n)$ [4]. Thus for T_1 spaces of sufficient size, $(R_0)(2, m, n)$, $(R_0)(3, m, n, p)$, etc. could be used to give many more topologically distinct topological properties all behaving in the same manner as $(R_0)(1, n)$.

Of all topological properties W such that $T_1 = (W$ and $T_0)$, weakly $R_0 = R_0$ is unique in that it is the only one for which weakly W exists.

Below the results above are used to give infinitely many new characterizations of T_2.

3. Infinitely Many New Characterizations of T_2

In the 1961 paper [1], the following characterizations of T_2 were given: For a space (X, T), the following are equivalent: (a) (X, T) is T_2, (b) (X, T) is R_1 and T_1, and (c) (X, T) is R_1 and T_0, which will be used along with the results above to give infinitely many new characterizations of the T_2 separation axiom.

Replacing Q in Theorems 2.1 and 2.2 above by R_1 gives the next two results.

Corollary 3.1. *Let* $\mathcal{S} = \{S\text{o} \mid S$ *is a topological property*, So *exists*, *and* So *implies* $T_2\}$. *Then* $(R_1)\text{o} = T_2 \in \mathcal{S}$, *for each weakly* Po *property* W *such that* Wo *implies* T_2, $($*weakly* $W\text{o})\text{o} \in \mathcal{S}$, *each element of* $\mathcal{S}$ *implies* T_2, $(R_1)_{min} = (R_1$ *or* "*not*-T_0"$)$ *is weaker than* R_1, $(R_1)_{min}$ *is the least element of* $\mathcal{S}$, *and* $(((R_1)_{min})$ *and* $T_0) = T_2$.

Corollary 3.2. *$(R_1)_{min}$ is the least topological property W such that $(W$ and $T_0) = T_2$ and, thus R_1 is not unique in the statement of Theorem* 1.1.

Replacing Q in Theorem 2.6 by R_1 gives the following new characterization of T_2.

Corollary 3.3. *$(R_1)(n) = ((R_1)$ or $T(n))$ is a topological property weaker than R_1 and stronger than $(R_1)_{min}$ such that a space has property T_2 iff it has property $((R_1)(n)$ and $T_0)$.*

Combining Theorem 2.7 with Corollary 3.3 gives infinitely many new characterizations of T_2 spaces.

Replacing Q in Theorem 2.8 by R_1 gives the following new characterization of T_2.

Corollary 3.4. *$(R_1)_{(min,max)} = (R_1$ and "not-T_0") is the least topological property weaker than T_2 and stronger than R_1 such that a space is T_2 iff it is $((R_1)_{(min,max)}$ and $T_0)$.*

Replacing Q in Theorems 2.11 and 2.12 by R_1 gives more new characterizations of T_2.

Corollary 3.5. *$(R_1)(1, n)$ is weaker than T_2 and stronger than $(R_1)_{(min,max)}$ and $T_2 = ((R_1)(1, n)$ and $T_0)$.*

Corollary 3.6. *For each T_2 space (Y, S), there are infinitely many topologically spaces all with topologically distinct topological properties that are weaker than T_2 and stronger than $(R_1)_{(min,max)}$, which together with T_0, equals T_2 and all having a T_0-identification space homeomorphic to (Y, S).*

For T_2 spaces of sufficient size, $(R_1)(2, m, n)$, $(R_1)(3, m, n, p)$, etc. could be used to give many more topologically distinct topological properties all behaving in the same manner as $(R_1)(1, n)$.

Of all topological properties W such that $T_2 = (W \text{ and } T_0)$, weakly $R_1 = R_1$ is unique in that it is the only one for which weakly W exists.

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