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USING THE WORKING BACKWARDS STRATEGY OF PROBLEM-SOLVING IN TEACHING MATHEMATICS TO FOSTER MATHEMATICS SELF-EFFICACY

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Abstract

In this study, we qualitatively explore how teaching problem-solving focusing on the working backwards strategy enhances students' efficacy beliefs to solve problems in mathematics. Efficacy-beliefs in one's capacity to organize and execute the courses of action required to produce given attainments constitute the best predictive motivational component of performance. Participants were 131 students of four junior-high school classes. The students came from average socioeconomic backgrounds. This qualitative case study was designed to elicit tacit knowledge on problem-solving self-efficacy of students pre- and post-using the working backwards strategy. A non-participant observation and five in-depth interviews were conducted in each class. Two discussions of the methods in the class were recorded. Data were analyzed using constant comparative analysis and grounded theory techniques. Results showed that teaching problem-solving focusing on the working backwards strategy enhanced students' problem-solving efficacy beliefs, self-regulation, and contributed to mathematical thinking performances.

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1. Introduction

Problem-solving has been a staple of school mathematics since the early 1980s. Its importance has been emphasized in documents that guide mathematics teaching and learning in various countries, and researchers have sought to better understand students' thinking and reasoning, to improve their problem-solving and, ultimately, their learning (Jiang et al. [16]). Most mathematics educators agree that the development of students' problem-solving abilities is a primary objective of instruction and how this goal is to be reached involves consideration by the teacher of a wide range of factors and decisions (Lester [19]).

More than four decades ago, Bandura [1] introduced the concept of self-efficacy beliefs in one's capacity to organize and execute the courses of action required to produce given attainments. Since that time, research in many arenas has demonstrated the power of efficacy judgement in human learning, performance, and motivation. Research on self-efficacy in the academic achievement arena is of particular importance to educators in general and especially to the learning of mathematics.

In this study, we attempt to teach the working backwards strategy of problem-solving, and diagnose students' self-efficacy beliefs pre- and post experiencing the strategy. While developing our case, we address the following aspects: perspectives on mathematics problem-solving instruction; the working backwards strategy, and the nature of efficacy beliefs' formation and change.

1.1. Perspectives on mathematics problem-solving instruction

All problem-solving research traditions agree that a problem is a task for which individuals do not know what to do to get an answer (Holth [15]). Researchers think that problem-solving should be both an end result of learning mathematics and the means through which mathematical concepts, processes, and procedures are learned. Both approaches have merit (DiMatteo and Lester [14]; Lester [19]). According

to a wider definition, successful problem-solving involves coordinating previous experiences, knowledge, familiar representations and patterns of inference, and intuition in an effort to generate new representations and related patterns of inference that resolve some tension or ambiguity that prompt the original problem-solving activity (Lester and Kehle [20]). Each of these ingredients should be attended to any program aimed at equipping prospective teachers with the proficiencies needed to teach mathematics either *for* or *via* problem-solving. Moreover, what is needed is to subsume problem-solving within a much broader category: "mathematical activity," and to give a prominent role to the meta-cognitive activity engaged in by the individual or group. This includes the act of comparing that occurs after conclusions are drawn and a solution is obtained. It might take place at any time and at any point during the entire process. This regular and continual monitoring of one's work is a key feature of success on complex mathematical tasks. The degree to which the individuals choose to compare their current state with earlier states can be considered a determinant of task complexity and, in fact, is the primary way to distinguish "routine" from "non-routine" problems (Lester [19]). More challenging, substantive problems demand more complex processes such as planning, selecting strategies, identifying sub-goals, choosing or creating appropriate representations, conjecturing, and verifying that a solution has been found. For non-routine tasks, a different type of perspective is required, one that emphasizes the making of new meanings through construction of new representations and inferential moves (Lester [19]).

Problem-solving is an important component of mathematics education as it is the single device that seems able to achieve three of the values of mathematics at school level: the functional, logical, and aesthetic values: Approaching mathematics through problem-solving can create a context that simulates real life and therefore justifies the mathematics rather than treating it as an end in itself (Taplin and Chan [37]). Researchers have emphasized the importance of problem-solving as a means of developing the logical thinking aspect of mathematics (Taplin

and Chan [37]). Another way in which a problem-solving approach is valuable is as an aesthetic form. Problem-solving allows the student to experience a range of emotions associated with various stages in the solution process. Mathematicians, who solve problems successfully, say that the experience of having done so contributes to their appreciation of the power and beauty of mathematics, the "joy of banging your head against a mathematical wall, and then discovering that there might be ways of either going around or over that wall" (Olkin and Schoenfeld [25], p.43). There is also the desire to be engaged with the task for a long time. However, although it is this engagement which initially motivates the solver to pursue a problem, it is still necessary for certain techniques to be available for the involvement to continue successfully.

The NCTM recommends that math programs include numerous and varied experiences with problem-solving as a method of inquiry and application. As well as developing knowledge, the students develop an understanding of when to use particular strategies. Students need to develop their own theories, test them, test the theories of others, discard them if they are not consistent, and try something else (NCTM [23]). Students are encouraged to discuss the processes which they are undertaking in order to improve understanding, gain new insights into the problem, and communicate their ideas (Taplin and Chan [37]). Solving of carefully selected problems helps to develop, refine, and cultivate creativity (Jiang et al. [16]; Novotna et al. [24]).

Thinking in a productive way requires the problem solver to interpret a situation mathematically, which usually involves progression through iterative cycles of describing, testing, and revising mathematical interpretations as well as selecting relevant quantities, identifying operations that may lead to new quantities, integrating, modifying, or refining sets of mathematical concepts drawn from various sources, and creating meaningful representations. It provides opportunities for students to elicit their own mathematics as they work the problem (Lesh and English [17]; Lesh and Zawojewski [18]).

Sriraman and English [32] provide a "modelling" approach to teaching problem-solving. Modelling problems are realistically complex situations where the problem solver engages in mathematical thinking beyond the usual school experience and where the products to be generated often include complex artifacts or conceptual tools that are needed for some purpose, or to accomplish some goal (Lesh and Zawojewski [18]). Modelling uses real-world contexts that draw upon several disciplines. Our world is increasingly governed by complex systems, which are becoming increasingly important in the everyday lives of both children and adults (Sriraman and English [32]). Educational leaders emphasize the need to develop students' abilities to deal with these systems for success beyond school. These abilities include: interpreting, describing, explaining, constructing, manipulating, and predicting complex systems, in which planning, monitoring, and communicating are critical for success (English [10], [11]).

Zollman [41] contends that some alternative approaches that have emerged in the last 25 years are not used today, as no single instructional method directly affects learning.

Many factors influence learning in the classroom: the curriculum, the student, the class, and the teacher. The various approaches will be successful if all influences are dedicated to student learning, knowledgeable of mathematics content, and skilled in implementation processes. The connections between problem-solving abilities and mathematical concepts Sriraman and English [32] want will need to include the school community, the curriculum, the methods, the teacher, the peers, as well as the individual student (Zollman [41]). Students' knowledge, beliefs, attitudes, emotions, and dispositions all play a part in determining what happens during instruction. Whichever approach is adopted, or if some combination of approaches is used, research is needed that focuses on the factors that influence student learning (Lester [19]).

The focus on teaching mathematical concepts through problem-solving contexts and inquiry-oriented environments is characterized by teachers helping students to construct a deep understanding of mathematical ideas and processes by engaging them in creating, conjecturing, exploring, testing, and verifying in mathematics (Lester et al. [21]). Specific characteristics of a problem-solving approach include: mathematical dialogue; teachers providing enough information to establish the background to the problem, and students clarifying, interpreting, and attempting to construct one or more solution processes, teachers encouraging students to make generalizations about rules and concepts — all these processes are central to mathematics thinking, and serve as a means for students to construct, evaluate, and refine their own mathematical theories and theories of others (Taplin and Chan [37]).

Teachers who subscribe to a teaching *via* problem-solving contexts and inquiry-oriented environments must be adept at designing and selecting good problems and activities, at listening to and observing students as they engage with problem-solving activities (Lester [19]); at asking the right questions, at knowing when to prod and when to withhold comment (Cai [5]; DiMatteo and Lester [14]); at keeping tasks appropriately problematic for students; at being able to plan and reflect (Lester [19]; Stephan and Whitenack [35]).

1.2. The working backwards strategy

Research on problem-solving strategies has revealed a great variety of strategies used by students, one of which is the working backwards (Burris [4]). Some problems ask learners to figure out what the starting point must have been. Working backwards is a strategy that works well for this type of problem. It is particularly useful when trying to discover proofs. Instead of starting from what you know and working toward what you want, start from what you want, and ask yourself what you need in order to get there (Rusczyk [30]). Athletes see themselves winning even before the competition begins. They visualize success. Engineers make drawings of finished products even before they know how to build them. According to Covey [7], highly effective people start with the end in mind.

Working backwards is the strategy of undoing key elements in the problem in order to find a solution. For example, a problem might tell the learner that after going to the store and buying milk for $2, Tony had $3 in his wallet. The problem asks how much money Tony must have had before he bought the milk. By working backwards, the learner uses the opposite or inverse of the operation that was performed. Since spending $2 is subtraction, one has to perform the inverse operation and add $2 back to the $3. Learners have figured out that Tony must have started with $5. An inverse operation undoes whatever the original operation did. When making a series of computations, one can start with data presented at the end of the problem and end with data presented at the beginning of the problem. This is called working backwards. Recording the work in list form facilitates review and identification of important steps that are necessary to reach a solution.

After students have completed their problem and have come up with a solution with which they are satisfied, reflecting on the problem-solving process is recommended. Much can be gained by taking the time to reflect and look back at what has been done, what worked, and what did not. This can help predict how to use the strategy as well as which additional strategies to use for solving future problems (Polya [29]).

Working backwards is a particularly useful problem-solving strategy when students can clearly define the goal or the end state of the problem, and know a sequence of operations that were used. Reversing the operations step by step helps problem-solvers to describe the initial conditions or the most efficient path to the goal state. Working backwards can be applied to two types of problems: logic problems, to which it is often applied, such as the famous one about crossing the river with a cabbage, a goat, and a wolf (The Math Forum [38]). In this problem, the goal state is known (everything on the opposite side of the river), as well as the legal moves (rowing one animal across the river leaving neither the cabbage nor the wolf with the goat). Students can

work backwards from the goal, asking themselves which animal must have been rowed across the river last. What must have happened just before that? The other type of problem to which the working backwards strategy can be applied involves operations on a quantity, in which students know the final outcome after all the operations have been applied, and need to find out the initial quantity. In this type of problem, it is useful to perform the inverse of each operation on the known end-quantity until one is left with the initial quantity. This inverse operation process is very similar to the process of solving an algebraic equation by "undoing" what has been done to the variable: e.g.,

$$a + b = c : c - a = b.$$

Many problem-solvers describe working backwards as playing a movie of the problem backwards in their heads. In reverse, giving becomes taking, doubling becomes halving, etc. When a description is agreed upon, it can be written down. It can then be worked backwards to figure out other ways of describing the problem, followed by discussions with others to reach a solution. Different strategies can be used to check the solution with or without using algebra. Teaching mathematics using this strategy can make mathematics a challenge for students (McDougal and Takahashi [22]). Creativity may be nurtured as their way of thinking and exploration. Teaching mathematics through problem-solving processes may develop students' self-confidence and pleasure, which can enhance motivation to work and think mathematically. Of all motivational components, self-efficacy is the most predictive component of student achievement.

1.3. The nature of efficacy beliefs' formation and change

In the last three decades, researchers have found links between student achievement and self-efficacy beliefs, and have developed a conceptual model to explain the formation and influence of efficacy beliefs in schools. Efficacy judgments are beliefs about individual capability, which might have consequences for the courses of actions they

choose to pursue, the effort they exert in those pursuits, and the ways in which they use the skills they possess. For example, Bouffard-Bouchard et al. [3] found that students with the same level of skill development in mathematics differed significantly in their math problem-solving success, depending on the strength of their efficacy beliefs. Students with a higher sense of self-efficacy more consistently and effectively applied what they knew than students with a lower sense of self-efficacy (Pajares [26]; Pajares and Graham [27]; Pajares and Miller [28]). Bandura [1, 2] postulated four sources of efficacy-shaping information: mastery experience, vicarious experience, social persuasion, and affective state:

A *mastery experience* is the most powerful source of efficacy information. The perception that a performance has been successful tends to raise efficacy beliefs, contributing to the expectation that performance will be proficient in the future, and vice versa; the perception that one's performance has been a failure tends to lower efficacy beliefs. Past successes build personal beliefs in the capacity of the individuals' self-efficacy, whereas failures tend to undermine the sense of self-efficacy.

A *vicarious experience* is one in which the skill in question is modelled by someone else. When a model with whom the observer identifies performs well, the observer's efficacy beliefs are most likely enhanced. When the model performs poorly, the observer's efficacy beliefs tend to decrease.

Social persuasion is another means of strengthening individuals' belief that they have the capabilities to set and achieve goals. Talks, workshops, professional development opportunities, and feedback about achievement can inspire action. Social persuasion alone may be limited in its power to create enduring changes in self-efficacy. However, when coupled with models of success and direct positive or negative experience, it can influence self-efficacy beliefs. The potency of persuasion depends on the credibility, trustworthiness, and expertise of the persuader (Bandura [1]).

Affective states, either anxiety or excitement, add to individuals' perceptions of self-capability or incompetence. Individuals with strong belief in their capability can tolerate pressure and crises and continue to function without debilitating consequences. Less efficacious individuals, however, are more likely to react dysfunctionally, which, in turn, increases the likelihood of failure. Thus, affective states may influence how individuals interpret and react to the myriad challenges they face.

Ultimately, the exercise of agency depends on how individuals interpret the information that shapes their efficacy beliefs. Bandura [1] emphasized that efficacy beliefs are created when individuals weigh and interpret their performance relative to other information. Changes in perceived efficacy result from cognitive processing of the diagnostic information that performances convey about capability. The same is true for all four sources. For some individuals, perceptions of self-efficacy arise from cognitive and metacognitive processing of the sources of efficacy-belief-shaping information described. Perceptions of self-efficacy provide a framework for understanding, predicting, and changing human behaviour. Self-efficacy theory maintains that all processes of psychological change operate through the alteration of the individual's sense of personal efficacy. Efficacy beliefs begin to form in early childhood when children are dealing with a wide variety of experiences, tasks, and situations. However, the growth of self-efficacy continues to evolve throughout life as people acquire new skills, experiences, and understanding (Bandura [1, 2]).

For junior-high school students who have just learned how to use equations to solve mathematical problems, demonstrating how problems can be solved without using equations but by using methods such as the working-backwards strategies, may enhance their creativity and develop mathematical thinking. We may not only achieve the functional, logical and aesthetic values of mathematics, but may also enhance students' self-efficacy beliefs to learn mathematics. This will foster better attainments, which will in turn, raise self-efficacy to learn mathematics.

The purpose of this qualitative case study is to teach problem-solving focusing on working backwards strategy, and to explore how this way of teaching fosters the formation of positive efficacy beliefs. We asked the following questions:

(1) What was the diagnosis of students' self-efficacy to solve problems pre-intervention?

(2) What was the diagnosis of students' self-efficacy to solve problems post-intervention?

2. Methodology

2.1. Participants

Participants were 131 students aged 12-13 years in four junior-high school classes. The students came from average socioeconomic backgrounds.

2.2. Design

This qualitative case study was designed to elicit, analyze, and describe tacit knowledge on students' self-efficacy beliefs to solve mathematics problems. The role of the researcher as a data collector was integral to the data that emerged, as humans are uniquely qualified for qualitative inquiry. This is due particularly to their ability to be responsive to the cues in the authentic situation, to collect information about multiple factors and across multiple levels, to take a holistic look at situations and try to reveal insights, to process data, and to ask for elaboration and clarification (Guba and Lincoln [13]). Throughout data collection and analysis, an effort was made to capitalize on these qualities to develop a rich perception of the factors that constructed the students' problem-solving self-efficacy. These methods attempted to present the data from the perspective of the observed participants, so that the researchers' cultural and intellectual bias did not distort the collection, interpretation, or presentation of the data. The qualitative

design consisted of systematic, yet flexible, guidelines for collecting and analyzing data to construct abstractions. The flexibility and openness of the qualitative approach enabled revelation of tacit knowledge on students' beliefs (Stake [34]).

Listening, observing, communicating, and remaining in the field of the study for a prolonged period allowed the researchers, first, to understand and create an authentic picture of the participants' thinking regarding their capability to solve problems, and second, to reflect deeply on their professional work as producers and enhancers of knowledge.

2.3. Research tools

We conducted one non-participant observation of class work and five in-depth interviews with informants in each class. We recorded the students' behaviour and comments during the discussions of the methods following experiences, and added their own reflective notes. This was done before and after the working backwards experience. The interviews contained open questions asking students to describe their feelings and thoughts regarding problem-solving, the atmosphere in class, the teacher, and about anything else related. Informative and elaborative questions were used when necessary. Communication was pleasant and the students were supportive. Each observation or interview lasted 40 minutes. We recorded two discussions of the methods that followed the experiences before and after problem-solving in each class. Atmosphere was relaxed.

2.4. Experiencing working-backwards

We used the strategy to teach algebra concepts in the classes for two weeks. Students $(n = 102)$ were supposed to solve problems by thinking and by working from the end to the beginning without using equations. This allowed students who were less competent in using linear equations to solve such problems. By creative thinking, one not only can solve a problem, but can also reach generalizations and extrapolation. The problems were solved either by the whole class or in pairs. The first problem was introduced as a game:

Problem A

The player who declares "1" is the loser

Two opponents compete in a natural numbers game, with the following rules:

(a) Each player chooses and declares a number.

(b) Any number can be picked. A number greater than 100 is recommended to lengthen the game.

(c) After one player has declared his/her chosen number, the second player has to choose and declare a number that is smaller, but bigger than or equal to half of the first player's number. For example, if the first player chooses 97, the opponent should choose a natural number $48 \leq n \leq 96$.

(d) The player who is obliged to declare "1" loses the game.

Two examples of this game are provided below:

Example of Game 1:			**Example of Game 2:**		
Player 1		Player 2	Player 1		Player 2
102	=>	68	120	=>	60
45	=>	25	38	=>	23
16	=>	10	20	=>	11
6	=>	4	8	=>	5
2	=>	1	3	=>	2
			1		
Player 1 is the winner			Player 2 is the winner		

Figure 1. Examples of two games.

How can one win this game?

As shown in the examples, the only mathematical knowledge required of the players is how to calculate the lower limit, which is half of the previous number, rounded up to the nearest whole number. Whoever reaches the number 2 first is actually the winner, because the opponent has only the number 1 left to choose.

When thinking from the end to the beginning, we reach the conclusion that the player who wants to choose number 2 on his turn needs, first, to choose number 5, which forces the other player to choose either 3 or 4. After several games and an analysis of the results (wins or losses), the strategic solution is reached: Choose a number from the series 2, 5, 11, 23 and work down the series to the next smallest number. What are the other numbers in the series when going from the smallest (2) to the larger numbers? The series 2, 5, 11, 23, 47, 95, 191, 383 is derived.

It is easy to see that every number in the series is 2 times the previous number plus 1. The player who chooses one of the numbers from this series and continues to choose the smaller number that is next to it in the series on each of his turns will definitely win.

The numbers in the series can be expressed using a recurrence relation.

When observing the numbers in the series, two interesting points are discovered:

(1) The series of its differences is a geometric series with a quotient of 2. {3, 6, 12, 24, 48, 96…}.

(2) The one place digits in the series, except for its first numbers, constitute a periodic sequence: …3, 1, 5, 7, 3, 1….

It is obvious that the strategic solution to this problem can be found only by thinking from the end to the beginning.

Problem B

Four customers entered an office supply store to buy pens, one after another. The first customer bought half the number of the pens in stock plus two more pens. The second customer bought half of the pens left in stock after the first customer's purchase plus two more pens. The third customer bought half of the pens left in stock after the second customer's purchase plus two more pens. The fourth customer had no more pens left to buy. How many pens had been in the store before any purchase was made?

Before considering the problem, it should be noted that the number of pens in the store before any purchase was made must have been an even number because half a pen cannot be purchased.

Algebraic beginning-to-end solution

Let x be the number of pens in stock before the customers made their purchases:

Table 1. The number of pens available prior to every purchase, the number of pens purchased, and the pens left in stock after the purchase was made

Customer number	Number of pens in stock prior to the purchase	Number of pens purchased by the customer	Number of pens remaining after the purchase
One	x	$0.5x + 2$	$0.5x - 2$
Two	$0.5x - 2$	$0.25x + 1$	$0.25x - 3$
Three	$0.25x - 3$	$0.125x + 0.5$	$0.125x + 3.5$

When the number of pens left after the purchase of the third customer is taken into account and is equated to zero, the equation is: $0.125x - 3.5 = 0$, and its solution is: $x = 28$. This is the standard method for solving problems of this type. If the student has not made a mistake in the algebraic calculation in any of the stages she/he will easily arrive at the answer and check its validity without using a calculator.

What happens if the student tasked with solving the problem is still unfamiliar with the algebraic expression and does not know how to solve linear algebraic equations, and certainly not equations of higher degrees? What if the question involves a larger number of customers, say seven, and only the seventh buyer has no pens left to buy at the store. In such a case, the table becomes very long and the chance of making a mistake in any one of the stages increases. At this point, we can introduce the working backwards strategy:

Algebraic end-to-beginning solution

According to the given in the problem, the third customer bought half of the pens left in stock after the second customer's purchase, plus two more pens. After the third customer's purchase, no pens were left in stock. In the other words, the number of pens left after the second customer's purchase needs to be calculated. A simple calculation shows that the number of pens remaining after the second customer's purchase is four because $0.5 \times 4 + 2 = 4$. Students sometimes go through some trial and error before reaching this conclusion.

In the next step, the number of pens remaining in stock before the second customer's purchase needs to be calculated. We already know that, after this purchase was made, there were four pens left for the third customer to buy. In this stage as well, after a simple calculation, it is discovered that the number of pens left in stock prior to the second customer's purchase was 12. In a similar manner, we turn our attention to the first customer and discover that the initial number of pens in the store was 28 and that 12 pens were left after the first customer's purchase. To sum up the steps from end to beginning:

The stock contained four pens before the third customer's purchase, 12 pens before the second customer's purchase, and 28 pens before the first customer's purchase. The relation between these numbers is

$$12 = 2 \times 4 + 4; \quad 28 = 2 \times 12 + 4.$$

Generalization

It can be generalized that this is a recurrence relation. To discover the quantity in the stock before the previous customer came to purchase pens, the number remaining for the current customer needs to be multiplied by two and then add four.

In the case of four customers, when each customer bought pens, 60 pens had to be available for the first customer to buy. In the same manner, in the case of five customers who bought pens, 124 pens had to be available for the first customer to buy. The end-to-beginning solution does not require the solving of an algebraic equation and allows for generalization with a larger number of customers.

Problem C: Robinson Crusoe "the second"

A sailor whose ship was wrecked in a storm washed up on a deserted island. While exploring his new home, he found a number of coconuts. To nourish himself, he began eating them: On the first day, he ate half of the coconuts, leaving half for the second day; on the third day, he had a quarter of the previous day's coconuts left; on the fourth day, he had a third of the previous day's coconuts left; on the fifth day, he had half of the previous day's coconuts left; on the sixth day, he had only one coconut left.

How many coconuts did the sailor find on the first day?

In this problem, the number of coconuts eaten by the sailor each day, relative to the number of coconuts left, is not fixed. The simplest strategy should be chosen to solve this problem. Let us use a table which presents the number of coconuts eaten each day and the number of coconuts left for the following days. Let x be the initial amount of coconuts:

Table 2. Days, number of coconuts left, and number of coconuts eaten each day

Day	Number of coconuts left	Number of coconuts eaten
Initial situation	x	
Day one	$\frac{1}{2}x$	$\frac{1}{2}x$
Day two	$\frac{1}{2}x - \frac{1}{4}x = \frac{1}{4}x$	$\frac{1}{2} \cdot \frac{1}{2}x = \frac{1}{4}x$
Day three	$\frac{1}{4}x - \frac{1}{16}x = \frac{3}{16}x$	$\frac{1}{4} \cdot \frac{1}{4}x = \frac{1}{16}x$
Day four	$\frac{3}{16}x - \frac{1}{16}x = \frac{2}{16}x = \frac{1}{8}x$	$\frac{1}{3} \cdot \frac{3}{16}x = \frac{1}{16}x$
Day five	$\frac{1}{8}x - \frac{1}{16}x = \frac{2}{16}x$	$\frac{1}{2} \cdot \frac{1}{8}x = \frac{1}{16}x$
Day six	0	$\frac{1}{16}x = 1$

It should be noted that this method requires familiarity with algebraic expressions and proficiency in simple algebraic operations, which are usually developed in junior-high schools and are therefore unsuitable for younger or less proficient students. Therefore, the problem needs to be approached alternatively, by using a clever solution integrating a beginning-to-end and an end-to-beginning solution, using a graphic illustration as demonstrated below:

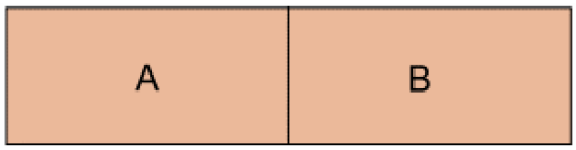

Figure 2. The initial number of coconuts and the number of coconuts left after the first day.

The area of the rectangle represents the initial number of coconuts. We divide the big rectangle into two smaller rectangles of equal areas A and B. Rectangle A represents the number of coconuts eaten by the sailor on the first day, and rectangle B represents the number of coconuts left after the first day.

Figure 3. The number of coconuts eaten on the second day, and the number of coconuts left at the end of the second day.

We divide rectangle B into two smaller rectangles of equal area: Rectangle C represents the number of coconuts eaten by the sailor on the second day, and rectangle D represents the number of coconuts left at the end of the second day. We then divide rectangle D into four rectangles of equal area: E, F, G, and H.

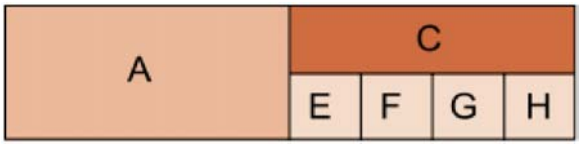

Figure 4. The number of coconuts eaten on the third, fourth, fifth, and sixth day.

Rectangle E represents the number of coconuts eaten by the sailor on the third day (a quarter of the remaining coconuts), rectangle F represents the number of coconuts eaten by the sailor on the fourth day (a third of the coconuts left after the third day), G represents the number of coconuts eaten on the fifth day (half of the coconuts left after the fourth day), and H represents the number of coconuts (only one) eaten on the sixth day.

The area of rectangle H is $\frac{1}{16}$ of the area of the original rectangle and represents the final coconut, which was eaten on the last day. It is then derived that, on his first day on the island, he found a total of 16 coconuts.

This strategy provides a visual demonstration of the calculation stages (presented in Table 2), in a way that young students or less proficient ones who are not familiar with algebraic operations can use – the working backwards.

Note: The task can be varied by making changes such as eating two coconuts on the sixth (last) day, dividing the days up differently, and even changing the number of days.

Problem D: The bunch of bananas and the monkey

The following problem is an example of a riddle of which every traditional "mathematical" attempt to solve it is bound to fail. Its solution is always welcomed with a smile. This problem belongs to the end-to-beginning type of problems and has more than one solution:

Three merchants who returned from a fair had a bunch of bananas, which they planned to divide between them. In addition, a monkey accompanied them the entire way. The merchants were tired after their long trip, and on arriving at the motel in the evening, they decided to go to sleep early and to divide up the bananas the next morning. At 22:00, one of the merchants woke up very hungry. He decided to eat his share of the bananas and to tell his friends the next morning. Before eating, he counted the bananas in the bunch and discovered that the number was not divisible by three. He gave one banana to the monkey, ate a third of the remaining bananas, and went back to sleep. At midnight (00:00), the second merchant woke up, not knowing that the first merchant had woken up before him. He, too, was hungry and decided to eat his share of the bananas and to tell the others in the morning. Before eating, he counted the bananas in the bunch and found that the number was not divisible by three. He gave one banana to the monkey, ate a third of the bunch, and went back to sleep. At 02:00, the third merchant woke up, not knowing that the other two merchants had woken before him. He, too, was hungry and decided to eat his share of the bananas and to tell his friends in the morning. He counted the number of bananas in the bunch and found that their number was not divisible by three. He gave one to the monkey, ate a third of the remaining bananas and went back to sleep. In the morning, when the three merchants woke up, they saw the tiny bunch of bananas that was left but none of them said a word. Each of them thought that only he had eaten the bananas. The three of them decided to divide what was left. They counted the number of remaining bananas and found that the number was not divisible by three. They gave one banana to the monkey and each took a third of the few remaining bananas. The question to be asked is: How many bananas were there at the beginning of the story?

Note that only one fact is certain; that the monkey received four bananas, one from each of the three merchants during the night, and another one in the morning.

End-to-beginning solution

(1) The first responses are usually as follows: "It is impossible to solve," "We are missing data," In actual fact, this problem has more than one solution!

(2) At this stage, most students try to solve the problem by guessing the number of bananas in the bunch when the merchants arrived at the motel. This leads them to form highly complex expressions (or equations for those who know algebra), which are impossible to solve.

(3) It is now the teacher's job to make sure that the students understand the arithmetic/mathematics problem at hand: the number of bananas that were in the bunch was N. N minus the number of bananas that the first merchant ate (n) was a multiple of 3 plus 1.

(4) $N = 3n + 1$ was also the situation after the first merchant gave a banana to the monkey and ate a third of the remaining bananas. The second merchant also found that the number of bananas left in the bunch was a multiple of 3 plus 1, as did the third merchant, as well as when they awoke in the morning. We are actually looking for a number (N) which is a multiple of 3 plus 1. If we subtract 1 and add one third of its sum, we get a number which is a multiple of 3 plus 1.

(5) After a certain amount of time, the students are instructed not to approach the problem conventionally, as they have been trying to do, but to try and begin from the end. To try to figure out how many bananas each merchant received in the morning, and then try and derive the number of bananas in each stage, one after the other, all the way back to the initial number of bananas in the original bunch, without, of course, forgetting the banana that the monkey received at each stage.

(6) To find the solution, we will begin from the smallest number of bananas that each merchant could have taken in the morning-one banana! According to this assumption, the bunch contained four bananas in the morning $(3 + 1)$, one was given to the monkey and each merchant

took one. If this was the case, then the four bananas in the bunch in the morning were two thirds of the bananas in the bunch before the third merchant ate his share at 02:00-six bananas. However, the monkey received one banana before the third merchant ate his share, and therefore, when the third merchant woke up, he found seven bananas in the bunch.

(7) This being the case, number 7 has to be two thirds of the bananas that were in the bunch before the second merchant had his share. This cannot be true, however, because 7 is an uneven number, and therefore cannot be two thirds of any other whole number.

(8) The next step is to start over again and assume that, in the morning, each merchant took two bananas. It follows that the bunch contained seven bananas in the morning $(3 \times 2 + 1 = 7)$, which was impossible, as explained in the previous step.

(9) The next attempt will be to check if it is the possible that each merchant took three bananas in the morning. In the other words, in the morning, the bunch of bananas contained 10 bananas, which will be proven impossible at a later stage. If the students continue to check the possibilities that each merchant took four, five, or six bananas, they will realize that each of these assumptions leads to a dead end. It becomes clear that the use of the trial and error strategy is inevitable in this case.

(10) The first result that leads to the correct solution is that each merchant took seven bananas in the morning. The bunch contained 22 bananas in the morning $(3 \times 7 + 1 = 22)$. In this case, 22 bananas are two thirds of the number of bananas that the bunch contained before the third merchant ate his share at 02:00. The third merchant found 34 bananas in the bunch when he woke up $(22 : \frac{2}{3} + 1 = 34)$, of which he gave one to the monkey and ate a third himself. In the same way, 34 bananas are two thirds of the number of bananas that were in the bunch before the second merchant ate his share. That means that the second

merchant found 52 bananas in the bunch $\left(34 : \frac{2}{3} + 1 = 52\right)$. Finally, the 52 bananas found by the second merchant when he woke up were two thirds of the bananas left in the bunch after the first merchant ate his share. In the other words, the first merchant found 79 bananas in the bunch $\left(52 : \frac{2}{3} + 1 = 79\right)$ when he woke up at 22:00 and the original bunch contained 79 bananas.

(12) Number 79 is only the minimal number for the solution to this problem, but this does not make it the only solution! Actually, every multiple of 7– the number of bananas taken by each merchant in the morning– will also lead to a solution. It is recommended to try other possibilities to arrive at additional solutions.

Summary of the problem

- In this problem, the students are introduced to an unconventional problem-solving method, which requires calculation and thinking that is not beyond the reach of weak students. The only requirement is knowledge of simple fractions, of how to find part of the whole, and the whole based on the part.
- This problem entails practice and internalization of the end-to-beginning solution.

This problem-solving method teaches the students that when they reach a dead end, or when data appear to be complicated or missing, a simple, alternative method can be tried, such as an end-to-beginning solution. After data collection and analysis, we discussed the results.

2.5. Data analysis

The qualitative methodological frame used for analysis was the criteria-oriented methodology. Of all qualitative frames, criteria-oriented methodology is closest to quantitative methodology. Data were analyzed using constant comparative analysis and grounded theory techniques (Charmaz [6]). The unit of analysis was a statement. The units were gathered under categories using three-stage coding: initial, axial, and

selective coding. Each unit was compared with other units or with properties of a category. The constant comparison of units resulted in a refined list of categories that were developed into conceptual abstractions called constructs. The qualitative analysis does not attempt to isolate variables but to form a system of constructs. Core constructs containing dense descriptions of evidence were formed. The whole system of constructs emerged only when repetition of the same constructs was obtained from multiple cases and when new units did not point to any new aspect (Charmaz [6]).

Data source triangulation (Stake [34]) was used to ensure the trustworthiness of the findings. When statements made by students in their interviews became evident in their actions, this resulted in triangulation. Moreover, the repetition of information several times in different interviews also resulted in triangulation. Member checks were employed to ensure the integrity of the study. The researchers' coding results were compared with those of an external rater (Denzin and Giardina [8]), and agreement was achieved by counting the number of sentences agreed upon (89%).

2.6. Ethics

All participants gave written informed consent to participate after being informed about the aims and methods of the study. Privacy was protected by using code numbers. The participants were offered the opportunity to read the results of the analysis if they so wished.

3. Results

Result 1.

All statements from observations, in-depth interviews, class discussions, and teacher reflections were qualitatively examined, gathered under criteria, and then under categories. The initial codes were provisional, comparative, and grounded in the data. During the three-step coding, the categories were redesigned (Strauss and Corbin [36]). The percentages of the evidence in each category were calculated and the following nine main themes emerged pre-experience:

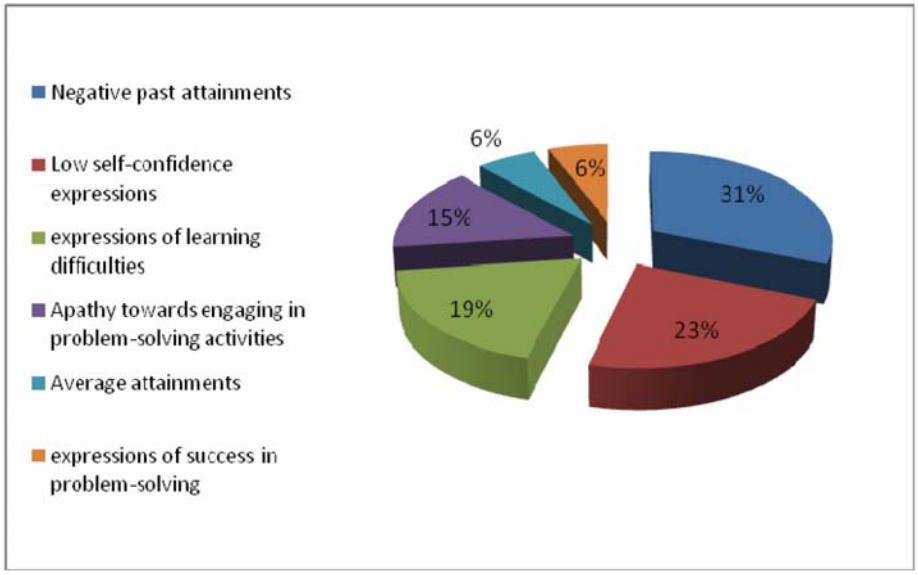

Figure 5. The students' self-efficacy profile to solve problems pre-experiencing the working backwards strategy.

According to Figure 5, the students were mostly worried about their future success in problem-solving. They have had negative past experiences in problem-solving, low self-confidence and difficulties, and they were apathetic about their practice (88%). This result answered Question 1.

Result 2.

Data were collected and analyzed again, using the same tools post experiencing the working backwards strategy. The following different expressions emerged:

(1) Mathematics thinking (20%).

(2) Mastery experience (13%).

(3) Self-confidence (13%).

(4) Control over problem-solving (13%).

(5) Pleasant feelings in mathematics lessons (12%).

(6) The working backwards strategy is an innovation (12%).

(7) Curiosity and challenge (8%).

(8) Social cooperation (5%).

(9) Persistence (4%).

The categories were refined and developed into conceptual abstractions called constructs (Charmaz [6]): Since mastery experiences (13%), pleasant feelings (12%), and social cooperation (5%) are self-efficacy sources, and curiosity and challenge (8%), and persistence (4%) are self-efficacy components, and since we found evidence of self-confidence to solve problems (13%), and control over problem-solving (13%), these categories were put together (68%) in one category called: self-efficacy to solve problems. In the same way, categories 1 and 6 were put together under the category of mathematical thinking (32%). This time learning difficulties were not reported. A theoretical map was constructed, and a two-construct self-efficacy profile finally emerged, as demonstrated in Figure 6:

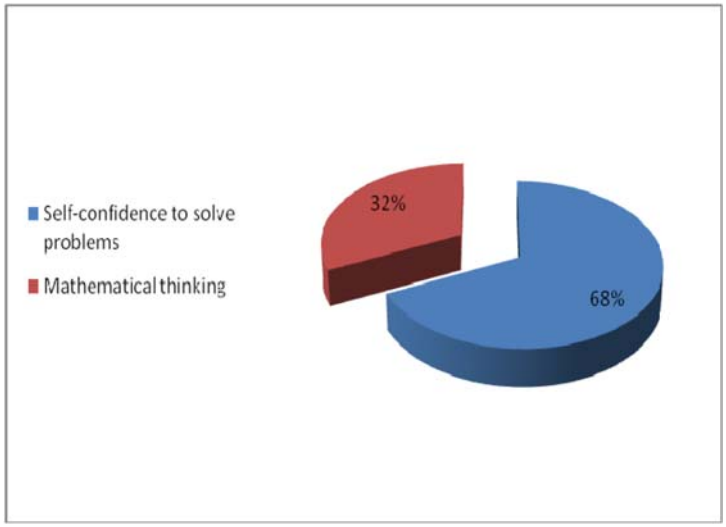

Figure 6. The students' self-efficacy profile to solve problems post-experiencing the working backwards strategy.

These results will be illustrated and discussed below.

4. Discussion

The results of this study show that teaching mathematics through the problem-solving working backwards strategy contributed to mathematics efficacy beliefs, and fostered mathematics thinking performances (see Figure 6).

4.1. The working backwards strategy fostered mathematics self-efficacy

Researchers believe that there is a problem with students' classroom experiences wherein students find little motivation to learn how to reason. Scholars argue that it is not appropriate merely to teach students how to reason, but it is important to build a case for students to learn how to reason. In the other words, we must make them feel the need to reason before teaching them how:

> "As usual, we were "stuck" in the regular way of solving problems." (Interview pre-intervention)

Current scholarly thinking in problem-solving research and teaching has thus focused on the need to improve students' attitudes and beliefs about the nature of mathematics, problem solving, and students' own personal competencies (Lester [19]; Lester and Kehle [20]; Tripathi [39]). One way of doing this was using the challenging strategy of working backwards. As students reached each step, they were challenged to go on to the next one, to reach the final solution. The working backwards strategy is composed of short episodes of success. When each episode was reinforced by the teacher or peers, and by the students' feelings of success, they gained the feeling of mastery, which encouraged them to go on. When the final solution was found, their self-efficacy was empowered, encouraging them to look for the next problem to solve from the end to the beginning. Enactive mastery experiences served as indicators of capability, as reflected in one students' words:

> "Mathematics doesn't seem so frightening after all." (Interview)

The most effective way of developing a strong sense of efficacy is through mastery experiences (Bandura [1, 2]). Performing a task successfully contributes to the personal expectation that performance will be proficient in the future (Dimopoulou [9]).

The results of this study show that students enjoyed these activities and felt pleasant positive feelings instead of anxiety and low self-confidence toward problem-solving. Positive feelings are another source of self-efficacy formation. Strong beliefs in capability can tolerate pressure and crises (Goddard et al. [12]). Indeed, perceptions of self-efficacy are negatively correlated with anxiety (Zimmerman [40]). Students were not afraid of word problems, and they did not give up or withdraw when encountering difficulties. Enjoyment and pleasant feelings fostered self-efficacy to think mathematically. The joy of problem-solving became solving problems. Students who experienced the joy of solving problems wanted to solve more problems:

> "It was fun to get to the solution; mathematics is not only equations but also logics." (Interview)

> "For the first time in my life, I felt comfortable; I'd like more of that." (Interview)

The choice of a problem-solving strategy suggests the first step and helps the problem solver to envisage subsequent steps.

Another source of self-efficacy formation is social persuasion, and this was obtained by facilitating peer and group interaction and sharing ideas, during problem-solving activities.

Due to the cooperative nature of the activities, the students were highly motivated. They requested additional time to devote to task D.

Students' discussions, in which they talked about their problem-solving ability, served as specific performance feedback from the teacher or a peer.

> "I showed it to friends. They did not expect it. They tried, and so we made it." (Interview)

> "The lesson dynamics was excellent and interesting." (Observation)

Although social persuasion alone might have limited power to create enduring changes in efficacy beliefs, action was inspired by talks, professional discussions, and feedback about achievement. When social persuasion is coupled with positive, direct experience and positive emotions, it can influence efficacy beliefs to overcome difficulties and take new paths (Bandura [1, 2]).

Using the working backwards strategy for problem-solving, three out of the four self-efficacy formation sources were activated.

This resulted in curiosity, interest, effort expenditure, initiation, challenge, and persistence, as reflected in the students' words:

> "Unbelievable! We couldn't imagine that this way was possible!" (Interviews)

> "We enjoyed it so much that we'd like to have some more of these." (Observation)

The Principles and Standards for School Mathematics describe problem-solving-based teaching as using interesting and well-selected problems to engage students. Good problems can inspire the exploration of important mathematical ideas, nurture persistence, and reinforce the need to understand and use various strategies, mathematical properties, and relationships (NCTM [23]). Introducing concepts through real life children's stories gave the students a purpose for persistence. The more we can integrate genuinely real-world problems into school mathematics, the better our chances of enhancing students' motivation and competencies in mathematical problem-solving (Sriraman and English [32]).

> "Math is full of life secrets you can discover." (Interview)

Self-efficacy plays a crucial goal in every phase of self-regulation (Bandura [1]; Schunk & Zimmerman [31]). Teaching mathematics using the working backwards problem-solving strategy contributed to students' self-regulated active involvement in the lesson. Students who were passive listeners in class became actively involved in work:

> "The good students started to use graphs of functions spontaneously as a possible problem-solving procedure; something that would not have occurred to them before." (Observation)

> "Students started to comment on their solving procedures, justify them, and in cases of written solutions, produced a verbal answer." (Observation)

Through this strategy, students clearly defined the goal states and operations, imagined reversing the operations, compared and checked their solutions in a variety of ways, discovered the mathematical relationships, similar to what is done with the guess and check strategy, but without having to guess (The Math Forum [38]). They found unknown initial conditions when goal states were known, and chose the most efficient from among multiple paths. They organized and kept track of complicated sequences. They described the problem context in terms of the quantities and relationships that could be worked backwards. They also transformed verbal descriptions into mathematical representations. They became active mathematics thinkers and performers.

> "The teacher shifts responsibility for the learning process of the next step, by helping the students develop the evaluation skill." (Observation)

By implementing this strategy, the students used self-regulatory skills, such as organizing, monitoring, evaluating, and regulating one's thinking processes (Zimmerman [40]). This is essential for successful problem solvers (Lester [19]). The discussion of the methods at the end of each step toward the final solution encouraged student reflection. Reflection involves investment of time and creative mental effort that culminates in reconstruction of knowledge and gaining new insights (Swartz [33]). Studies have shown that reflection enhances meta cognitive processes, which constitute self-regulated learning (Schunk and Zimmerman [31]). We have focused on developing the working backwards strategy and the ability to apply it to unfamiliar situations. We have focused on gathering, organizing, interpreting, and communicating information, formulating key questions, analyzing and conceptualizing problems, defining them, and developing curiosity, all of which are recommended by professionals (NCTM [23]).

4.2. The working backwards problem-solving strategy fostered mathematics thinking

All four problems were successfully solved by the students, as described by the observers:

Problem A. In the first games, the students started and the teacher won. However, after the first four/five games, the students understood that whoever chooses number 5 on his turn wins the game. After one or two more games, the students realized that whoever chooses number 11 on his turn, and then chooses number 5 on the next turn, wins. The average students (according to their teachers' assessment) took about 25 minutes from the start of the activity to reach a generalization regarding the numbers that should be chosen in order to win the game. The more able students took about 20 minutes from the start of the activity to realize how to win the game.

Problem B. The average and low level students used the trial and error beginning-to-end strategy, assuming that the number of pens the store had in stock before any purchases were made was an even number. After about 10 minutes from the start of the activity, one or two pairs of students found the correct number and demonstrated how they calculated the number of pens bought by each of the customers. After about 20 minutes, the rest of the students could solve the problem.

The more proficient students used linear algebraic equations, and the methods of the pairs of students were closely observed. About 50% of the students tried to solve the problem algebraically as described in the solution to the problem, not all of them succeeded, and about 30% of the students used the trial and error beginning-to-end strategy. About 20% of the students tried to solve the problem by using an end-to-beginning strategy.

Five minutes after the start of the activity, 6% of the students found the end-to-beginning solution. About 10 minutes after the start, the first pair of students found the solution using the trial and error strategy, and

only after about 15 minutes from the start, the first pair found a solution algebraically. This shows that it sometimes takes longer to find the algebraic solution than when using the alternative strategies.

Problem C. The proficient students first solved the problem in about eight minutes, using the end-to-beginning strategy. The rest of the students solved it using the same strategy in 15 minutes. Afterwards, they were asked to solve the problem algebraically. The first pairs solved it in about 10 minutes and the rest of the students in about 25 minutes. This was because some of them had made some mistakes in the algebraic fraction operations.

The less proficient students needed more time to solve the problems due to the operations with fractions. The first pair took 11 minutes to find the solution and the last pair took 22 minutes.

Problem D. The students could not find the solution and the teacher directed them to work step by step using the working backwards end-to-beginning strategy, while analyzing and discussing problems that arose during their mathematical thinking. The observer did not measure the time, but this problem included dealing with many more mathematical concepts and lasted more than one lesson.

We have demonstrated the importance of solving problems by working from the end to the beginning. The most natural way to solve problems A and D was to use the end-to-beginning strategy. Problems B and C could be solved from the beginning to the end or the other way round. The first method requires an algebraic equation and the second method allows a solution based on thinking alone. A variety of strategies allows choice of the most efficient one for the appropriate problem and even to find new or alternative original avenues.

The results of this study show that teaching through the working backwards problem-solving strategy required students' mathematics thinking performances. When students encountered a problem for which they had no immediately apparent solution, nor an algorithm that they could directly apply to get an answer, they must have read the problem

carefully, analyzed it for whatever information it had, and examined their own mathematical knowledge to see if they could come up with a suitable strategy. The process forced the reorganization of existing ideas and the emergence of new ones as students worked on problems with the help of the teacher, who acted as facilitator by asking questions that helped students review their knowledge and construct new connections. The result was an enrichment of the network of ideas through understanding. This was how the students came to view mathematics as a discipline founded on reasoning. In this process, students developed habits of the mind that enabled them to analyze other situations that they might encounter in life, whether mathematical or otherwise:

> "… then we got to the solution! Now we can solve other big problems." (Observations)

> "Some students used the regular way but it took more time and they got confused with their equations." (Observations)

> "You have to be creative, there is always another way; math develops your mind." (Interview)

Literature on problem-solving in mathematics has extensively discussed the need to teach students to reason mathematically. In mathematics, problem-solving is not a content strand but a pedagogical stance, when students are exposed to a variety of tasks that require them to collate and analyze previous knowledge and yet offer a challenge. Problem-solving is thus seen as a means of developing students' reasoning skills (Tripathi [39]). By using the working backwards problem-solving strategy, we encouraged the students' divergent thinking and creativity. By asking students to offer both critique and explanations, we developed their argumentation skills, so that students did not just have to reason but actually believed that such reasoning is an intrinsic aspect of mathematics.

The working backwards strategy for problem-solving can contribute significantly to mathematics education. Not only is it a means for developing logical thinking, it can provide students with a context for acquiring mathematical concepts, can enhance the transfer of skills to unfamiliar situations and is an aesthetic form in itself. It can provide

students with a tool for constructing their own ideas about mathematics and for taking responsibility for their own learning. The challenge for teachers, at all levels, is to develop the process of mathematical thinking alongside the knowledge and to seek opportunities to present even routine mathematics tasks in problem-solving contexts. It is also an opportunity to empower students' positive self-efficacy to be engaged in mathematical learning and thinking, which encourages students to challenge the next assignment.

> "Only now I realized that every profession has math in it." (Interview)

5. Conclusions and Implications

We witnessed the growth of students' self-efficacy beliefs as they became more self-regulated in problem-solving, and vice versa. When self-regulatory processes play an integral role in the development and use of study skills, students become more acutely aware of improvements in their academic achievement and experience a heightened sense of personal efficacy (Zimmerman [40]). Efficacy beliefs focused attention on the students' beliefs about the effectiveness of their chosen learning methods. The self-regulatory approach to problem-solving provided students with the opportunities to see how activities under their control could bring them rewarding feelings. By having students attend to increasing feelings of self-efficacy, we helped instill in them a strategic orientation to thinking skill development in mathematics learning. At the end of their experience, they said the following:

> "We had mathematics lessons without mathematics. We used our heads, not equations!" (Interview)

> "Math is a beauty. It's endless. Anyone can make it." (Observations)

In this context, Bandura [2] avers that students' early mathematical years are extremely important because it is during these years that their self-efficacy begins to form. We believe that the working backwards strategy should be encouraged at elementary stages, when there is, clearly, a strong need to focus on incorporating a culture of reasoning.

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Endnotes

More illustrations of detailed data is available by request.

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