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APPLICATIONS OF NEW TOPOLOGICAL PROPERTIES IN THE STUDY OF SINGLETON SET, INDISCRETE, AND CONTINUOUS IMAGE PROPERTIES

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Abstract

Within a recent paper, many new topological properties were introduced giving many new properties and tools for use in the continued study of topology. In this paper, the new properties and tools are used in an in depth investigate of the singleton set, indiscrete, and continuous image properties.

1. Introduction and Preliminaries

Within a recent paper [1], many new properties and examples were given, providing many new, topologically fundamental properties and tools that have not only greatly expanded the study of topology, but, also, in a meaningfully manner, impacted current studies in topology, and will continue to do so in the future.

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Definition 1.1. Let P be a topological property. Then "not-P" is the negation of P, provided the negation exists [1].

Theorem 1.1. *$L = (T_0$ or "not-T_0") is the least of all topological properties* [1].

Within this paper, L will be used to denote the least topological property.

Theorem 1.2. *For a topological property P, the following are equivalent*: (a) *"not-P" exists*, (b) *"not-P" is a topological property, P is stronger than L, and $P \neq$ "not-P"*, (c) *$P \neq L$ and $P \neq$ "not-P"*, (d) *P is stronger than L, and* (e) *"not-P" is stronger than L* [1].

Theorem 1.3. *Let P be a topological property different from L. Then $L = (P$ or "not-P")* [1].

In two follow up papers; [2] and [3], the new properties given above proved to be useful and needed in answering several questions that naturally arise in the study of topology. However, since the existence of the least topological property L had not even been considered prior to the paper cited above [1], its existence has created problems. Within the paper [2], it was proven that L is a product property and that every product space exhibits the product property L, which is far from the intent of the creators of product properties and created a disconnect in the study of product properties. Thus, to maintain continuity between past studies of product properties and any future studies of product properties, a correction was needed in the definition of product properties [2]. Likewise, the existence of L created a problem in the study of subspace properties, leading to a needed change in the definition for subspace properties [3]. Within the paper on subspaces [3], the continued investigation of the new properties as related to subspaces led to an unexpected discovery that motivated the following question.

Question 1.1. Is there a space that simultaneously exhibits all subspace properties and, if so, give such a space together with a proof?

Within that paper, singleton set topological spaces played an important role in answering new, never before asked subspace questions and it was proven that singleton set topological spaces simultaneously satisfies all subspace properties. It was well-known that singleton set spaces satisfies many properties, including properties that are not subspace properties, but the connection between singleton set spaces and subspace properties was unknown prior to the new subspace property paper [3]. Within this paper, singleton set spaces and several other properties continue to be investigated by using the new properties and tools.

2. The Singleton Set and the Indiscrete Properties

Definition 2.1. A space (X, T) has singleton set property iff X is a singleton set. Clearly, the singleton set property is a topological property. In the past, singleton set spaces have been used as simple examples of topological spaces, but in this paper, singleton set spaces are given greater importance and status as is deserved.

Theorem 2.1. *Let* (X, T) *be a space. Then* (a) (X, T) *has singleton set property iff* (b) *for each set* Y *and each topology* U *on* X *and each topology* V *on* Y, $f : X \to Y$ *is onto iff* $f : (X, U) \to (Y, V)$ *is a homeomorphism.*

Proof. (a) implies (b): If, as stated in the theorem, $f : (X, U) \to (Y, V)$ is a homeomorphism, then $f : X \to Y$ is onto. Thus consider the case that $f : X \to Y$ is onto. Since X is a singleton set and $f : X \to Y$ is onto, Y is a singleton set and the only topology on each of X and Y is the singleton set topology. Hence $f : (X, U) \to (Y, V)$ is a homeomorphism.

(b) implies (a): Suppose X is not a singleton set. Let x and y be distinct elements of X, let $Y = X$, let U be the discrete topology on X, let V be the indiscrete topology on Y, and let $f : X \to Y$ be the identity function. Then $f : X \to Y$ is onto, but $f : (X, U) \to (Y, V)$ is not a homeomorphism, which is a contradiction. Hence (b) implies (a).

Definition 2.2. A space (X, T) has the indiscrete property iff T is the indiscrete topology on X.

Clearly, the indiscrete property is a topological property. As in the case of singleton set spaces, within this paper indiscrete spaces is given greater recognition and status.

Theorem 2.2. *A space* (X, T) *has singleton set property iff* (X, T) *is* T_0 *and has the indiscrete property.*

Proof. If (X, T) has the singleton set property, then, clearly T is the indiscrete topology on the singleton set X and (X, T) is T_0.

Conversely, suppose (X, T) is T_0 and has the indiscrete property. Suppose X is not a singleton set. Let x and y be distinct elements of X. Since (X, T) has the indiscrete property, then every open set containing x contains y, but, since (X, T) is T_0, there exists an open set containing only one of x and y, which is a contradiction. Thus (X, T) has the singleton set property.

As given above, each new topological property will, in fact, give many additional topological properties. Examples in this case of additional topological properties from the work above include “not-singleton set property”, “not-indiscrete property”, (“singleton set property” and compact), (regular and “not-indiscrete property”), and (“not-singleton set property” and “not-indiscrete property”).

Subspace properties have been long studied in classical topology [5]. As given above, the existence of the least topological property L created a disconnect in the study of subspace properties, which is corrected in the definition below.

Definition 2.3. Let P be a topological property. Then P is a subspace property iff $P \neq L$ and for a space (X, T), (X, T) has property P iff each subspace of (X, T) has property P [3].

Theorem 2.3. *The indiscrete property is a subspace property.*

Proof. Since $L \neq$ (the indiscrete property), then the indiscrete property is a possible candidate for a subspace property.

Suppose a space (X, T) has the indiscrete property. Assume there is a subspace (Y, T_Y) of (X, T) that does not satisfy the indiscrete property. Then Y contains two or more elements and then exists a T_Y-open set U that is a proper subset of Y. Let O be a T-open set such that $U = Y \bigcap O$. Then $O \in T$ and O is a proper subset of X, which is a contradiction. Thus every subspace of (X, T) has the indiscrete property.

Let (X, T) be a space such that every subspace of (X, T) has the indiscrete property. Since (X, T) is a subspace of itself, then (X, T) has the indiscrete property.

Combining Theorem 2.3 with the fact that T_0 is a subspace property gives the next result.

Corollary 2.1. *The singleton set property is a subspace property.*

Within the paper [3], it was established that for each subspace property P, "not-P subspace property" is not a subspace property and for each subspace property Q for which (Q and "not-P") exists, (Q and "not-P") is not a subspace property. Applying these results to the results above immediately gives many more topological properties that are not subspace properties.

Product spaces with the Tychonoff topology were introduced in 1930 [4]. As in the case of subspace properties, the existence of L created a disconnect in the study of product properties that led to the definition below [2]. In this paper, all product spaces will have the Tychonoff topology.

Definition 2.4. Let P be a topological property. Then P is a product property iff $P \neq L$, and a product space has property P iff each factor space has property P [2].

Theorem 2.4. *The indiscrete property is a product property.*

Proof. Since the indiscrete property is not L, the indiscrete property is a candidate for a product property.

Suppose (X, W) is the product space of the topological spaces $\{(X_\alpha, T_\alpha) \mid \alpha \in A\}$ and (X, W) has the indiscrete property. Assume there exists a $\beta \in A$ such that (X_β, T_β) does not have the indiscrete property. Let $O_\beta \in T_\beta$ be a nonempty proper open set. For each $\alpha \in A$, $\alpha \neq \beta$, let $O_\alpha = X_\alpha$. Then $\prod_{\alpha \in A} O_\alpha$ is a nonempty proper open set in X, which is a contradiction. Thus each factor space of (X, W) has the indiscrete property.

Conversely, suppose each factor space (X_α, T_α), $\alpha \in A$, of the product space (X, W) has the indiscrete property. Assume (X, W) does not have the indiscrete property. Let $\mathcal{O}$ be a nonempty proper open set in (X, W). Let $\{x_\alpha\}_{\alpha \in A}$ be an element of $\mathcal{O}$. For each $\alpha \in A$, let $O_\alpha \in T_\alpha$ such that $x_\alpha \in O_\alpha$, $O_\alpha = X_\alpha$ except for finitely many $\alpha \in A$, $\prod_{\alpha \in A} O_\alpha \in W$, and $\prod_{\alpha \in A} O_\alpha \subset \mathcal{O}$. Since $\mathcal{O}$ is a proper nonempty subset of X, there exists a $\beta \in A$ such that O_β is a nonempty proper subset of X_β, which is a contradiction. Hence (X, W) has the indiscrete property.

Combining Theorem 2.4 with the fact that T_0 is a product property gives the next result.

Corollary 2.2. *The singleton set property is a product property*.

Within the paper [2], it was established that for each product property P, "not-P product property" is not a product property and for each product property Q for which (Q and "not-P") exists, (Q and "not-P") is not a product property. Applying these results to the results above immediately gives many more topological properties that are not product properties.

3. The Continuous Image Property

In classical studies of topology, for a topological property P, the question of whether the continuous image of each space with property P must have property P has been asked and resolved for many properties. However, as above, the existence of L was never a consideration in such investigations and, as given below, does create a disconnect in the continuous image question.

Since every space (X, T) has property L [2] and every topological property P implies L, then, regardless of what property P that (X, T) has, (X, T) would exhibit the L continuous image property, which is far from the intent and expectations of continuous image properties and led to the definition below.

Definition 3.1. Let P be a topological property. Then P has the continuous image property iff $P \neq L$ and for each space (X, T) with property P, each continuous image of (X, T) has property P.

Theorem 3.1. *The continuous image property is a topological property*.

The straightforward proof is omitted.

Theorem 3.2. *The singleton set property is a continuous image property.*

Proof. Let (X, T) be a space with the singleton set property. Let (Y, S) be a continuous image of (X, T) and let $f : (X, T) \to (Y, S)$ be continuous and onto. Then $f : X \to Y$ is onto and by Theorem 2.1, $f : (X, T) \to (Y, S)$ is a homeomorphism and (Y, S) has the singleton set property.

Since compact is a continuous image property and is not the singleton set property, then the converse of Theorem 3.1 is false.

Theorem 3.2. *The singleton set property is the strongest continuous image property.*

Proof. Let (Y, S) be a singleton set space with $Y = \{y\}$. Let P be a continuous image property. Let (X, T) be a space with property P. Let $f : X \to Y$ be the constant function $f(x) = y$ for all $x \in X$. Then $f : (X, T) \to (Y, S)$ is continuous and onto and (Y, S) has property P. Since the singleton set property is a continuous image property, then the singleton set property is the strongest continuous image property.

Theorem 3.3. (a) *P be a continuous image property that implies T_0 iff* (b) *P is the singleton set property.*

Proof. (a) implies (b): Suppose there exists a continuous image property P that implies T_0 that is not the singleton set property. Let $Y = \{u, v\}$, $u \neq v$, and let S be the indiscrete topology on Y. Then (Y, S) has property ((indiscrete property) and "*not* - T_0"). Let (X, T) be a space with property P. Let x be an elements of X and let $f : X \to Y$ defined by $f(x) = u$ and $f(y) = v$ for all $y \in X$, $y \neq x$. Then $f : (X, T) \to (Y, S)$ is continuous and onto and (Y, S) has property P, which implies T_0, but then (Y, S) has property (((indiscrete property) and "*not* - T_0") and T_0), which is a contradiction. Thus, if P is a continuous image property that implies T_0, then P is the singleton set property.

(b) implies (a): Let P be a singleton set property. Then P is a continuous image property. Since T_0 is a subspace property, then P implies T_0.

Corollary 3.1. *If P is any one of the subspace properties T_i, $i = 0, 1, 2$, Urysohn, 3, or $3\frac{1}{2}$, then P is a continuous image property iff P is the singleton set property.*

Theorem 3.4. *The indiscrete property is a continuous image property.*

Proof. Let (X, T) be a space with the indiscrete property. Let (Y, S) be a continuous image of (X, T) and let $f : (X, T) \to (Y, S)$ be continuous and onto. Suppose (Y, S) does not have the indiscrete property. Let U be a nonempty, proper open set in Y. Then $f^{-1}(U)$ is open in X and, since T is the indiscrete topology, $f^{-1}(U) = X$, but then $f^{-1}(Y \setminus U)$ is a nonempty subset of X and $((f^{-1}(U)) \cap (f^{-1}(Y \setminus U))) = \phi$, which is a contradiction. Thus, the indiscrete property is a continuous image property.

Within the product property paper [2], for each product property P, P and "not-P" were used to instantly give many new topological properties that are not product properties. In the subspace property paper [3], for each subspace property P, P and "not-P" were used to instantly give many new topological properties that are not subspace properties. Thus, the question of whether for a continuous image property P, can P and "not-P" be used to instantly give new topological properties that are not continuous image properties arises? Below, this question is addressed.

Definition 3.2. Let Q be a topological property. Then Q is not a continuous image property iff there exists a space (X, T) with property Q and a continuous image (Y, S) of (X, T) that is not Q, i.e., (Y, S) has property "not-Q".

Theorem 3.5. *Let P be a continuous image property. Then "not-P" is not a continuous image property.*

Proof. Suppose there is a continuous image property P for which "not-P" is a continuous image property. Then singleton set property is the strongest continuous image property and singleton set property implies P are true statements, but then singleton set property implies "not-P" is a false statement, which contradicts "not-P" is a continuous image property. Thus "not-P" is not a continuous image property.

Hence each continuous image property instantly gives a not continuous image property. Within the paper [2], it was proven that the set of product properties is closed under finite intersections. In the paper [3], it was proven that the set of subspace properties are closed under finite intersections. Thus, the question of whether the set of continuous image properties is closed under finite intersections arises and is addressed below.

Theorem 3.6. *Let P and Q be continuous image properties. Then (P or Q) is a continuous image property.*

Since (P or Q) exists, the proof is straightforward and omitted.

Theorem 3.6 can be combined with the Principle of Mathematical Induction to show the set of continuous image properties is closed under finite unions.

Theorem 3.7. *Let P and Q be continuous image properties. Then $P \neq$ "not-Q".*

Proof. Suppose there exist continuous image properties P and Q for which P = "not-Q". Let (X, T) be a space with property "not-Q" and (Y, S) be a continuous image of (X, T) that has property Q. Since P = "not-Q", then (Y, S) has property P, but since Q = "not-P", (Y, S) is "not-P", which is a contradiction. Hence $P \neq$ "not - Q".

Theorem 3.8. *Let P and Q be continuous image properties. Then* (P *and* Q) *exists and is a continuous image property*.

Proof. Suppose there exist continuous image properties P and Q for which (P and Q) does not exist. Then P is strictly stronger than “not-Q”, Q is strictly stronger than “not-P”, and “not-Q” = (P or (“not-P” and “not-Q”)). Since $P \neq$ “not-Q”, then “not-Q” = (“not-Q” and “not-P”) = “not-(Q or P)” and Q = (P or Q), but then (P and Q) = (P and (P or Q)) = ((P and P) or (P and Q)) = P, which is a contradiction. Therefore, for continuous image properties P and Q, (P and Q) exists. Then by a straightforward argument, which is omitted, (P and Q) is a continuous image property.

Theorem 3.8 can be combined with the Principle of Mathematical Induction to prove the set of continuous image properties is closed under finite intersections. Thus many more continuous image properties are quickly and easily obtained.

Theorem 3.9. *Let P and Q be continuous image properties such that* (P *and* “*not-Q*”) *exists. Then* (P *and* “*not-Q*”) *is not a continuous image property*.

Proof. Suppose there exist continuous image properties P and Q such that (P and “not-Q”) exists and (P and “not-Q”) is a continuous image property. Then Q and (P and “not-Q”) are continuous image properties, but (Q and (P and “not-Q”)) does not exist, which is a contradiction. Hence, for continuous image properties P and Q for which (P and “not-Q”) exists, (P and “not-Q”) is not a continuous image property.

Therefore many more not continuous image properties are instantly known.

References

[1] C. Dorsett, Weakly P corrections and new, fundamental topological properties and facts, Fundamental Journal of Mathematics and Mathematical Sciences 5(1) (2016), 11-20.

[2] C. Dorsett, Pluses and needed changes in topology resulting from additional properties, submitted.

[3] C. Dorsett, New properties, tools and changes for subspace properties and singleton set spaces, submitted.

[4] A. Tychonoff, Uber die topologishe Erweitarung von Raumen, Math. Ann. 102 (1930), 544-561.

[5] S. Willard, General Topology, Addison-Wesley Pub., Co., 1970.

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