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# $L^p$-SOLUTIONS TO $x'' - a(t)x = 0$

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**Abstract**

In this note, we give sufficient conditions on $a(\cdot)$ for the $L^2$-solution to the limit point operator $L[x] = x'' - a(t)x = 0$ to be an $L^p$-solution for all integral $p = 1, 2, \ldots$. Specifically, we show that $a(\cdot)$ needs to be just continuous, positive, and bounded from below.

## 1. Introduction

In this note, we examine the bounded solutions to the limit point operator

$$L[x] = x'' - a(t)x = 0 \quad (t \geq 0,\ a(t) > a_0 > 0, \text{ and } a(\cdot) \,\varepsilon\, C[0, \infty)). \tag{1}$$

We shall show that not only is this solution in $L^2[0, \infty)$, but that it is also in $L^p[0, \infty)$ for $p = 1, 2, 3, 4, 5, \ldots$, that is for all positive integers. The

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other linear independent solution will be shown to be unbounded. Here we are dealing with a class of limit point operators (see [1, Chapter 9] for a discussion of limit point and limit circle operators and [6] for its relationship to the spectral theory of differential operators; for some other interesting results about limit point operators see [2]-[4]). By definition, a limit point operator is one having only one linearly independent solution to (1) in $L^2[0, \infty)$.

Statement and proof or our main result now follows.

## 2. Main Result

**Theorem.** *Given the differential equation* (1). *Then there exists a bounded solution* $x(\cdot)$ *to* (1) *such that* $x(\cdot)$ *is an element of* $L^p[0, \infty)$ *for* $p = 1, 2, 3, \dots,$ *and, furthermore, both* $x(t) \to 0$ *and* $x'(t) \to 0$ *as* $t \to \infty$.

**Proof.** We will first show that $x(t)$ is in both in $L^1[0, \infty)$ and $L^2[0, \infty)$. The first step is to construct an unbound solution $z(t)$ to (1) with initial conditions $z(0) = 1$ and $z'(0) = 1$. Obviously, $z(t)$ is unbounded since $z''(t) > 0$ for $t > 0$ which implies $z(t) > z'(0)t + z(0) = t + 1$. Next, we write the other linearly independent solution to (1) as $x(t) = z(t)\int_t^\infty ds / z(s)^2$ (for a derivation see [5, pp. 87-91]). We may rewrite this as $x(t) = \int_t^\infty (ds / z(s)^2) / (1 / z(t))$. Notice that our hypotheses imply that $1/z(t)$ is square integrable since $1/z(t)^2 < (t+1)^{-2}$. Next, apply L'Hospital's rule to $x(t)$ as $t \to \infty$. The result is $x'(t) = 1 / z'(t)$ which approaches 0 as $t \to \infty$ since $z'(t) = z(0) + \int_0^t a(s)z(s)ds$ approaches $\infty$ as $t \to \infty$ because $z(t)$ is unbounded and $a(t) > a_0 > 0$.

We now show that $x'(t) \to 0$ as $t \to \infty$. First, note that $x(t)$ can never be 0 by construction. Also, $x'(t)$ can never equal 0. Should $x'(t) = 0$ at some point $t_1$ then we have $x''(t_1) > 0$ so $x(t)$ starts increasing and becomes unbounded which contradicts the boundedness of $x(t)$. Consequently, since $x(t)$ is bounded, monotonic, and never 0, $x'(t)$ must approach 0 as $t \to \infty$.

It is obvious that $\int_0^\infty x''(t)dt = \int_0^\infty a(t)x(t)dt = -x'(0)$ and $\int_0^\infty x'(t)dt = -x(0)$. This proves both $x(\cdot)$ and $x'(\cdot)$ are elements of $L^1[0, \infty)$. By multiplying Equation (1) by $x(t)$ and then integrating the first term of (1) by parts from 0 to $t$ yields

$$x(t)x'(t) - x(0)x'(0) - \int_0^t x'(s)^2 ds - \int_0^t a(s)x(s)^2 ds = 0. \quad (2)$$

Finally, letting $t \to \infty$ and rewriting (2), we immediately see that

$$\int_0^\infty x'(s)^2 ds + \int_0^\infty a(s)x(s)^2 ds = -x(0)x'(0). \quad (3)$$

We have now shown that both $x(\cdot)$ and $x'(\cdot)$ are in both $L^1[0, \infty)$ and $L^2[0, \infty)$. We complete the proof by proving $x(\cdot) \in L^p[0, \infty)$ for $p > 2$ using induction. Assuming that $x(t) \in L^p[0, \infty)$, we will show that $x(t) \in L^{p+1}[0, \infty)$. To do this, multiply (1) by $x(t)^p$ and then integrate the first term of (1) by parts from 0 to $t$ to obtain

$$x^p(t)x'(t) - x(0)^p x'(0) - \int_0^t px'(s)^2 x^{p-1}(t)ds - \int_0^t a(s)x(s)^p ds = 0. \quad (4)$$

Letting $t \to \infty$ and rewriting (4), we now have

$$\int_0^\infty px'(s)^2 x^{p-1}(t)ds + \int_0^\infty a(s)x(s)^p ds = -x(0)^p x'(0), \quad (5)$$

which complete the proof.

**Remark.** Note that the condition $a(t) > a_0 > 0$ is needed to guarantee that the solution is an $L^p$-solution for all $p$ since this may not be true if just $a(t) > 0$. Specifically, consider the equation

$$x'' - x / (4(t+1)^2) = 0. \tag{6}$$

The bounded solution to (6) is $x(t) = b(t+1)^{-1/2(\sqrt{2}-1)}$, where $b$ is any real non-zero constant. This solution, however, is not in $L^1[0, \infty)$ or $L^2[0, \infty)$. On the other hand, the bounded solution to

$$x'' - 2x / (t+1)^2 = 0 \tag{7}$$

is $x(t) = b(t+1)^{-1}$ ($b$ is a constant) which is not in $L^1[0, \infty)$ but is an element of $L^2[0, \infty)$. The general bounded solution to

$$x'' - kx / (t+1)^2 = 0 \tag{8}$$

is $x(t) = b(t+1)^{1/2 - 1/2\sqrt{(4k+1)}}$. Only for values of $k > 2$ are the solutions $L^p$-solutions.

## 3. Conclusion

What we have essentially shown that under the stated hypotheses the bounded solutions to the limit point operator (1) are not only $L^2$-solutions but $L^p$-solutions as well.

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