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THE 1-2-3-EDGE LABELLING AND VERTEX COLOURS

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Abstract

A labelling of the edges of a graph is called vertex-colouring if the labelled degrees of the vertices yield a proper colouring of the graph. In this paper, we show that such a labelling is possible from the label with numbers of the set $\{1, 2, 3\}$ for all graphs not containing components with exactly 2 vertices.

Keywords: edge-labelling, vertex-colouring, ordering algorithim, labelling algorithm, complete graph, latest edge, layer.

1. Introduction

All graphs in this note are simple and the numbers of vertices are finite. For notation is not defined here we refer the reader to [4]. For some $k \in \mathbb{N}$, let $f : E(G) \to \{1, 2, \cdots, k\}$ be an integer labelling of the

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edges of a graph $G = (V(G), E(G))$, which have n vertices. This labelling is called edge labelling if the labelled degrees $S_i := \sum_{j=1, j \neq i}^{n} f(v_i v_j)$ for all integer $1 \leq i \leq n$ of the vertices yield a proper vertex-colouring of the graph; (i.e., S_i's the sum of the edge labels of vertex v_i, where $1 \leq i \leq n$ that it is a vertex colour of v_i). It is easy to see that for every graph which does not have a component isomorphic to K_2, there exists such a labelling for some integer positive k. In 2002, Karonski et al. (see [6, 10]) conjectured that such a labelling with $k = 3$ is possible for all such graphs ($k = 2$ is not sufficient as seen for instance in complete graphs and cycles of length not divisible by four). At first constant bound of $k = 30$ was proved by Addario-Berry et al. [1], which was later improved to $k = 16$ in 2008 by Addario-Berry et al. [2], $k = 13$ by Wang and Yu [11]; and to $k = 6$ by Kalkowski et al. [8]; and Kalkowski et al. introduced the best bound for $k = 5$ in [9].

In this paper, we show a completely different approach by introducing an algorithm for labelling that improves the bound to $k = 3$, that it is finally value for k in the all graphs which are simple and finite. The following theorem is the main result of this paper.

Theorem 1.1. *For every graph G without components isomorphic to the complete graph* K_2, *there is a labelling* $f : E(G) \to \{1, 2, 3\}$, *such that the induced vertex weights* $S_i := \sum_{j=1, j \neq i}^{n} f(v_i v_j)$ *properly colour* $V(G)$.

Conjecture 1.2. As above notation, there are distinct numbers of S_i's, $1 \leq i \leq n$, of a graph G of order n, for a 1-2-3-edge labelling and vertex colours.

Conjecture 1.3. For all graph G of order n, there are the $\chi(G)$ numbers of S_i's, $1 \leq i \leq n$, with this 1-2-3-edge labelling and vertex colours, where $\chi(G)$ is the number colours of the vertices on the graph G.

2. The Method and Algorithms

In this section, we introduce two algorithms on a graph $G = (V, E)$ of order n. At the first algorithm, we present a vertex ordering on a connected components of G, which we call it the *ordering algorithm* and in continue in the second algorithm we label the edges which their vertices be named by the ordering algorithm and this is the labelling algorithm. We now express the first one.

2.1. The ordering algorithm

At first we order all vertices according to their degrees with a descending sequence as follow:

$L_1 = \{u_1^{(1)}, u_2^{(1)}, \dots, u_{n_1}^{(1)}\}$, $L_2 = \{u_1^{(2)}, u_2^{(2)}, \dots, u_{n_2}^{(2)}\}, \dots, L_t = \{u_1^{(t)}, u_2^{(t)}, \dots, u_{n_t}^{(t)}\}$

in which $\deg(u_i^{(j)}) = a_j$ for $j = 1, 2, \dots, t$ and $i = 1, 2, \dots, \max\{n_1, n_2, \dots, n_t\}$. We call L_k as the K-th layer, so the degree of all of vertices in K-th layer is a_k, it is trivial that $\sum_{j=1}^{t} n_j = n$. We have $\mathbf{1 \leq a_t} < \cdots < \mathbf{a_2 < a_1 \leq n - 1}$. We want to establish an one-to-one corresponding between $L_{j+1} = \{u_1^{(j+1)}, u_2^{(j+1)}, \dots, u_{n_{j+1}}^{(j+1)}\} \leftrightarrow \{v_{n_1+n_2+\ldots+n_j+1}, v_{n_1+n_2+\ldots+n_j+2}, \dots, v_{n_1+n_2+\ldots+n_{j+1}}\}$. In the first step, we select an arbitrary vertex in L_1 and call it as v_1 and if there exist the another vertex in L_1 and it is adjacent to v_1 we call it v_2. Otherwise (i.e., there exist some vertices in L_1 and they are not adjacent to v_1) we select one of them arbitrary and call it as v_2. In continuation, we find a vertex in L_1 like $u_r^{(1)}$ by the following priority and we call it v_3,

(–) $u_r^{(1)}$ is adjacent to v_1 and v_2,

(–) $u_r^{(1)}$ is adjacent to v_1 and it is not adjacent to v_2,

(–) $u_r^{(1)}$ is adjacent to v_2 and it is not adjacent to v_1,

(–) $u_r^{(1)}$ is not adjacent to v_1 and v_2, we call $u_r^{(1)} = v_3$, respectively. And go on this process till all of the vertices in L_1 are named as $v_1, v_2, \ldots, v_{n_1}$.

In the second layer, we continue the above process with L_2 in other words we select a vertex in L_2 like $u_s^{(2)}$ with the following priority and call it v_{n_1+1},

(–) $u_s^{(2)}$ is adjacent to $v_1, v_2, \ldots, v_{n_1}$,

(–) $u_s^{(2)}$ is adjacent to $v_1, v_2, \ldots, v_{n_1-1}$ and it is not adjacent to v_{n_1},

$\vdots$

(–) $u_s^{(2)}$ is not adjacent to $v_1, v_2, \ldots, v_{n_1}$,

in each of the above cases we set $u_s^{(2)} = v_{n_1+1}$, respectively. And go on this process till all of the vertices in L_2 are named as $v_{n_1+1}, v_{n_1+2}, \ldots, v_{n_1+n_2}$. Finally, each of the vertices in the all of layers are named and established the following one-to-one corresponding:

$$\{u_1^{(1)}, u_2^{(1)}, \ldots, u_{n_1}^{(1)}\} \leftrightarrow \{v_1, v_2, \ldots, v_{n_1}\},$$

$$\{u_1^{(2)}, u_2^{(2)}, \ldots, u_{n_2}^{(2)}\} \leftrightarrow \{v_{n_1+1}, v_{n_1+2}, \ldots, v_{n_1+n_2}\},$$

$$\{u_1^{(t)}, u_2^{(t)}, \ldots, u_{n_t}^{(t)}\} \leftrightarrow \{v_{n_1+n_2+\ldots+n_{t-1}+1}, v_{n_1+n_2+\ldots+n_{t-1}+2}, v_{n_1+n_2+\ldots+n_t}\},$$

in which $n_1 + n_2 + \ldots + n_t = n$.

For example, we consider the following graph and apply the ordering algorithm to call its vertices (see Figure 1).

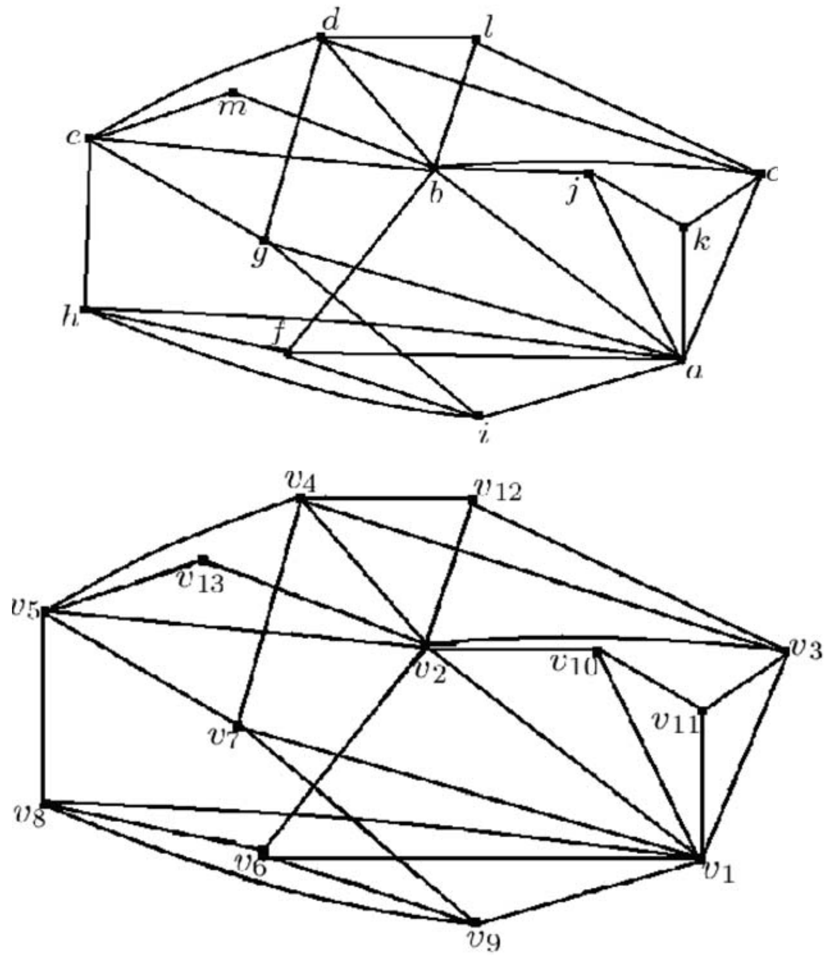
The vertices of degree of $8 : L_1 = \{a, b\} = \{u_1^{(1)}, u_2^{(1)}\}$.

The vertices of degree of $5 : L_2 = \{c, d, e\} = \{u_1^{(2)}, u_2^{(2)}, u_3^{(2)}\}$.

The vertices of degree of $4 : L_3 = \{f, g, h, i\} = \{u_1^{(3)}, u_2^{(3)}, u_3^{(3)}, u_4^{(3)}\}$.

The vertices of degree of 3 : $L_4 = \{j,\ k,\ l\} = \{u_1^{(4)},\ u_2^{(4)},\ u_3^{(4)}\}$.

The vertices of degree of 2 : $L_5 = \{m\} = \{u_5^{(1)}\}$.



The vertices of degree 8, $L_1 = \{v_1, v_2\}$.
The vertices of degree 5, $L_2 = \{v_3, v_4, v_5\}$.
The vertices of degree 4, $L_3 = \{v_6, v_7, v_8, v_9\}$.
The vertices of degree 3, $L_4 = \{v_{10}, v_{11}, v_{12}\}$.
The vertices of degree 2, $L_5 = \{v_{13}\}$.

Figure 1. Solution of ordering algorithm for example in this figure.

So in the first step we have $v_1 = a, v_2 = b,$ the next step since c is adjacent to v_1 and v_2 so $v_3 = c$ also in this layer there exist d, e but none of them are not adjacent to v_1, they are adjacent to v_2, therefore we set $v_4 = d$ or $v_4 = e$, etc.

At the end of this process, we have

$v_1 = a, v_2 = b, v_3 = c, v_4 = d, v_5 = e, v_6 = f, v_7 = g, v_8 = h, v_9 = i,$

$v_{10} = j, v_{11} = k, v_{12} = l, v_{13} = m$ (see Figure 2).

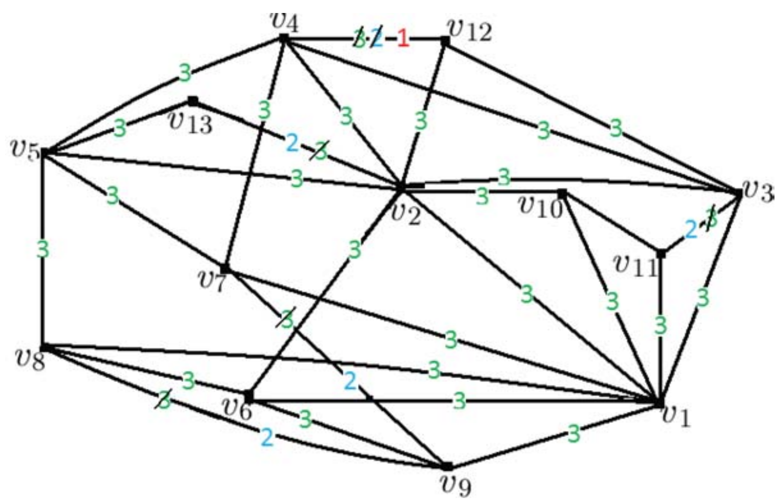


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Figure 2. Solution of labelling algorithm for example in Figure 1.

Definition 2.1. For a vertex $v_i (1 \leq i \leq n)$ the latest edge $v_i v_j$ is an edge of the graph G such that $\deg(v_j) \leq \deg(v_k), \forall v_i v_k \in E(G)$.

2.2. The labelling algorithm

After the run of ordering algorithm, we assign the number 3 to all of edges that incident to v_1, and calculate S_1. Then continue with vertex v_2, and assign the number 3 to each edge that it incident to v_2 and it has not labelled before, then calculate S_2. If $v_1 v_2 \notin E$ or $S_1 \neq S_2$, then go to vertex v_3. Otherwise $v_1v_2 \in E$ and $S_2 = S_1$, in this case we subtract one unit from the latest edge of v_2 that has label 3, so $S_2 = S_1 - 1$ and continue with vertex v_3, i.e., we assign 3 to all of edges with endpoint v_3 that they were not labelled before.

If $v_1v_3 \in E$ and $S_3 = S_1$, then as before we subtract one unit from the latest edge of v_3 which it has not the label 1. Similarly if $v_3v_2 \in E$ and $S_2 = S_3$, we do the same process then we have $S_3 = S_2 - 1$.

We continue this process for vertex v_i, $4 \leq i \leq n-1$. Label every edge of v_i that is not labelled with 3. If there exist an index $j(1 \leq j < i)$ such that $v_iv_j \in E$ and $S_i = S_j$, that subtract one unit from the latest edge of v_i, that is labelled with number other one, and do as above. In the other words, we do calculate the new S_i by verify $f(v_iv_m)$ such that $v_iv_m \in E$ for $i < m \leq n$.

Suppose that there is no such vertex v_m, $i < m \leq n$, adjacent with vertex v_i such that the edge v_iv_m have a label 1. It means that $\forall m$, $i < m \leq n$, if $v_iv_m \in E$ we have $f(v_iv_m) = 1$. Then find the first vertex v_l before of the vertex v_i, where $i > l \geq 1$, and subtract one unit from the latest edge of the vertex v_l and check the condition of v_i with the vertices adjacent and verify S_l. Now, we start the process for the vertex v_{l+1} without changed the edge labelling and sum of the labelling for all vertices v_i, $1 \leq i \leq l$ and continue this process as above until the last vertex v_{n-1}. As an example see the following Figure 3, and finally Figure 4.

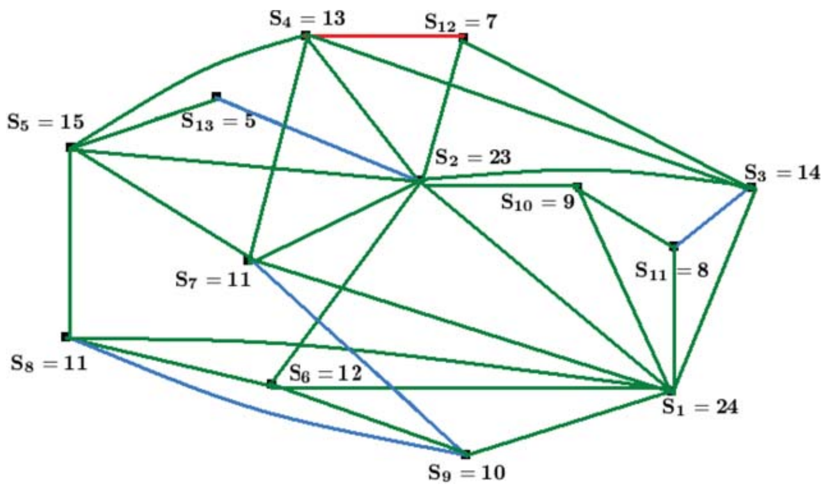

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Figure 3. The 1-2-3-edge labelling and vertex colours of Figure 1.

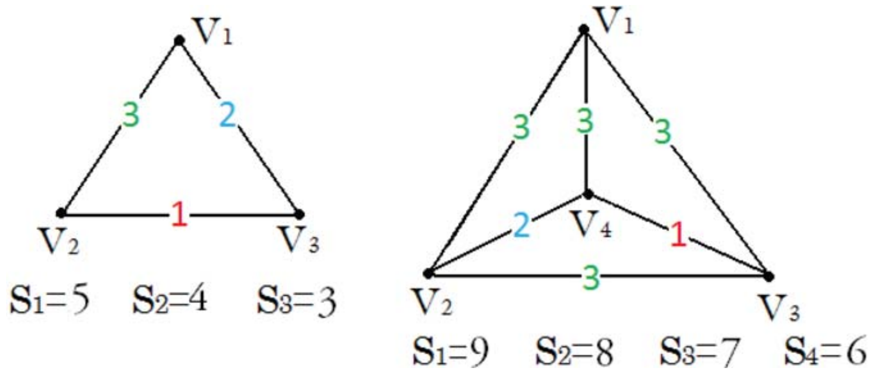

Figure 4. The 1-2-3-edge labelling for graph $K3$ and $K4$ by the special algorithm on K_n.

In the end of this section, we exhibit the following definition:

Definition 2.2. A subset of the vertices of a graph G that have the same degree a_i is called a layer i.

Definition 2.3. Let $G = (V, E)$ be a graph. The number of error of the vertex v of G which we denote by ER_v, is the number of adjacent vertex of v that are labelled before v, and also sum of their labels is equal to S_v in the self step of the algorithm.

Definition 2.4. Let $G = (V, E)$ be a graph with $|V| = n.$ For a vertex v of G that is i-th in the ordering of the algorithm, define X_v to be the number of vertices that are adjacent and before of v.

It is easy to see that $X_v \leq i - 1$ and $X_v \leq \deg(v)$, and also $ER_v \leq X_v$ for every vertex v.

In the next section, we first prove the algorithms in the special case of the complete graph K_n.

3. Special Case of the Algorithm on K_n

That is trivial that the run of the algorithms in a regular graph specially in the complete graphs is harder than to any other graphs, because these graphs have one layer. In the following lemma, we will show that the algorithms are correct for the complete graph K_n.

Lemma 3.1. *There exists an edge-labelling with numbers* 1, 2, *and* 3 *for the complete graph* K_n *with n vertices such that the sum of the labels in all vertices is different.*

Proof of Lemma 3.1. First we express the algorithm for K_n, because in the next section we will use the lower bound in the general case. These algorithms calculate the maximum sum that there exist for all vertices of the graph G. Indeed this lower bound is for the regular graphs, which is the harder case and here we have only one layer. We will prove that the lower bound of S_n in the complete graph K_n is $S_n \geq n + 2$.

3.1. Algorithm for graph K_n

Since all vertices of K_n are adjacent with together and they have the same degree $n-1$, so we don't need to run the ordering algorithm, we consider the second algorithm only. Mark the vertices with numbers $1, 2, \ldots, n$, we introduce the modified version of the above algorithm for the complete graph K_n.

Label every edges of v_1 with 3, so $S_1 = 3n - 3$. For the vertex v_2, label every edge that is not labelled before, with 3, and label edge v_2v_n with 2, and then we have $S_2 = 3n - 4$. For the vertex v_3, label every edge as above and label edge v_3v_n with 1, so $S_3 = 3n - 5$. For the vertex v_4, labelled every edge that is not labelled, with 3 and label edge v_4v_n with 1, and label edge v_4v_{n-1} with 2, so $S_4 = 3(n-1) - 3$.

For $4 \leq i \leq n-1$, we continue this process for every vertex v_i, in which we subtract one unit from the latest edges (which are not equal to 1) so that $S_i = S_{i-1} - 1$. Now we consider two cases.

- If i is even, that is $i = 2m$, then set $v_iv_n = v_iv_{n-1} = \cdots = v_iv_{n-m+1} = 1$ and label other edges with 3. Now, if $S_i \neq S_{i-1}$ continue this process for v_{i+1}, otherwise when $S_i = S_{i-1}$, then set $v_iv_{n-m} = 2$, and we have $S_i = S_{i-1} - 1$.

- If i is odd, that is $i = 2m+1$, then we set $v_iv_n = v_iv_{n-1} = \cdots = v_iv_{n-m+1} = 1$ and $v_iv_{n-m} = 2$; and label other edges with 3. Now if $S_i \neq S_{i-1}$, we go to the vertex v_{i+1} and we continue as above. If $S_i = S_{i-1}$, then we set $v_iv_{n-m} = 1$, and hence $S_i = S_{i-1} - 1$.

Finally, we obtain $S_1, S_2, \ldots, S_n$, for $v_1, v_2, \cdots, v_n$. Therefore, all edges of the complete graph K_n labelled with 1, 2, and 3 and having the desired property. In fact, for the graph K_n, we find n series with finite sequences of numbers 1, 2, and 3. In the other words, for every $i, 1 \leq i \leq n$ set $S_i := \sum_{j=1}^{n} f(v_i v_j)$ such that $f(v_i v_i) = 0$ and $f(v_i v_j) \in \{1, 2, 3\}$ for $i \neq j$. Then

$$(n-1) \leq S_n < S_{n-1} < \cdots < S_2 < S_1 \leq 3(n-1).$$

So it is sufficient to show that we can find such labelling for integer n such that $S_n = \min\{S_i\}_{i=1}^{n} \geq n-1$.

Now we will show that $S_n \geq n+2$, where $n \geq 4$, and this means that "we can label one edge of the vertex v_n with 3, one edge with 2 and $n-3$ edges with 1".

For $n = 1, 2$, there is no such edge-labelling, so $n \geq 3$. Also for the set of size less than 3, such an edge-labelling not exist. But for $n = 3$, we will have the edge $v_2 v_3$ with label 1 and the edge $v_1 v_3$ with 2 and the edge $v_1 v_2$ with 3. Since for $n \geq 4$; we have $n - 1 < n + 2 \leq S_n < S_{n-1} < \cdots < S_2 < S_1 \leq 3(n-1)$, it is sufficient to show that $S_1 - S_n \leq 2n - 6 < 2n - 1 \Rightarrow S_2 - S_n < 2n - 5 < 2n - 3$.

Now we prove this by induction on n. If $n = 4$, then G is the complete graph K_4, $S_4 = 6 \geq 4 + 2$ and $S_2 - S_4 = 8 - 6 < 3 = 2 \times 4 - 5$, see Figure 3.

Now suppose that it holds for the complete graph $K_{n-1} = (V', E')$. Hence $S'_2 - S'_{n-1} \leq 2(n-1) - 5 = 2n - 7$, where that S'_i is sum of the edge labels of $v_i S'_i = \sum_{j \neq i} f(v_i v_j)$ and $v_i v_j \in E'$ and $f(v_i v_j) \in \{1, 2, 3\}$.

Finally, we are ready to label the complete graph K_n, and calculate each S_i for all vertices $v_i (1 \leq i \leq n)$, by running the algorithm. So we have $S_1 = 3n - 3$, $S_2 = 3n - 4$, and so on.

Remove the vertex v_1 of the graph K_n. Suppose that R_1 is the sum of labels of vertex u_1 in the subgraph K_{n-1}, which vertex u_1 is the vertex v_2 on the graph K_n. So we will have $R_i = S_{i+1} - 3$, and $R_1 = S_2 - 3 = (3(n-1)-1) - 3 = 3n - 7$, such that for the vertex u_1, $R_1 = S_1' - 1$. Since $v_2 v_n = 2$ in the graph K_n, add the number 1 to the label of the latest edge $u_1 u_{n-1}$ of the vertex u_1, and modify $R_1 = (3n-7)+1$.

For the vertex u_2 add one number to its latest edge $u_2 u_{n-1}$, because $R_2 = S_2' - 1$. So the new sum is $R_2 = S_3 - 3 + 1 = 3n - 7$.

On the other hand, for the vertex v_{n-1}, since $R_{n-1} = S_n - 3$, we can add 2 numbers to two edges of K_{n-1}, so that we have $S_{n-1}' < S_n$. Thus for adding two edges to it we have $R_{n-1} = S_n - 3 + 2 = S_n - 1$.

The relations $R_2 = S_3 - 2$ and $R_{n-1} = S_n - 1$; imply that

$$2(n-1) - 5 = 2n - 7 \geq R_2 - R_{n-1} = S_3 - 2 - S_n + 1.$$

Since by run of the labelling algorithm on the K_n, $S_3 = S_2 - 1$, $S_2 - 3 - S_n + 1 = S_2 - S_n - 2 \leq 2n - 7$. Therefore $S_2 - S_n \leq 2n - 5$. The proof is now complete for K_n.

We encourage the readers to see other references [3-5] on the edge-labelling and Vertex-coloring of the complete graph K_n.

4. Main Results: The Proof of Theorem 1.1

We prove that, there is an edge-labelling with number 1, 2, and 3 for any connected component $G = (V, E)$ with n vertices of a graph H, such that sum of the edge-labels on adjacent vertices are not equal.

It is obvious that for each vertex v_i, of degree a_i for $1 \leq i \leq n$, the sum of S_i is at least a_i, because there exist the label one and this implies that the algorithm is convergence by the following proof.

We do this labelling with the mentioned algorithms in Section 2. In fact, we will show that the algorithms are convergence for each vertex v_i for integer $i(1 \leq i \leq n)$. In the first, we order the vertices with their degrees in a descending sequence by the ordering algorithm. This labelling algorithm runs on vertex v_1 and edge-labels with number 3. So $S_1 = 3a_1$ in which $a_1 = \deg(v_1)$.

Now run the algorithm for the vertex v_2. If the vertex v_2 is co-layer with v_1; (i.e., $\deg(v_2) = a_1$) and v_1, v_2 are adjacent $(v_1v_2 \in E)$, reduce one unit from the latest edge of v_2, and $S_2 = 3a_1 - 1 \neq S_1$. and if $v_1v_2 \notin E$, $S_2 = 3a_1$. Also if the vertex v_2 is not co-layer with v_1, then $\deg(v_2) = a_2 \neq a_1$, and $S_2 = 3a_2$.

Suppose that the algorithm runs for the vertex $v_3, \cdots, v_{i-1}$, for each integer i, and then obtain $S_3, \ldots, S_{i-1}$. Now we show that it acts for any vertex v_i. Suppose that $\deg(v_i) = d_i$, that v_i is in the layer r, where $d_i = a_r$. So these vertices are adjacent with X_i or X_{v_i} vertex v_1 until v_{i-1}. Thus, it is adjacent with $(d_i - X_i)$ numbers of the vertices v_{i+1} until v_n that are not coloured and edges are not labelled. Suppose that Y_i be the sum of labels of X_i preceding edges of v_i, that is, $Y_i = \sum_{j=1}^{i-1} f(v_iv_j)$. Since the labels are from numbers 1, 2, and 3, we have

$$(\mathrm{I}) : X_i \leq Y_i \leq 3X_i.$$

Also from Definition 2.4,

$$(\mathrm{II}) : ER_i \leq X_i \leq d_i.$$

Hence with respect to the algorithm labels all of $(d_i - X_i)$ remaining edges have number 3. So

$$S_i = 3(d_i - X_i) + Y_i = 3d_i + Y_i - 3X_i.$$

Now (I) implies that

$$3d_i + X_i - 3X_i \le S_i = 3d_i - X_i + Y_i \le 3d_i + 3X_i - X_i \le 3d_i + 2X_i.$$

It follows that,

$$3d_i - 2X_i \le S_i = 3d_i - X_i + Y_i.$$

Now from (II), we have

$$3d_i - S_i \le 2X_i \le 2d_i \Rightarrow d_i \le S_i \le 3d_i.$$

For the above inequality, we can show that the labelling algorithm holds for v_i, $1 \le i \le n-1$. This means that for all vertex v_i the labelling algorithm return S_i in the favorite bound, such that for all $v_iv_j \in E$, $f(v_iv_j) \in \{1, 2, 3\}$ and $S_i \ne S_j$.

When $i = n$, we have $Y_n = \sum_{j=1}^{n-1} f(v_iv_j)$ and

$$a_t = d_n = X_n \le n-1,$$

and

$$Y_n = S_n = 3(d_n - X_n) + Y_n.$$

Thus (I) implies that

$$d_n \le Y_n = S_n \le 3d_i = 3\deg(v_n).$$

Now the proof of Theorem 1.1 is complete.

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