Journal of Mathematical Sciences: Advances and Applications
Volume 38, 2016, Pages 49-71
Available at http://scientificadvances.co.in
DOI: http://dx.doi.org/10.18642/jmsaa_7100121635

COMMON PROPERTIES OF TRAPEZOIDS AND CONVEX QUADRILATERALS

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Abstract

In this paper, we show some interesting properties of the trapezoid which have to do with the perpendiculars that issue from the point of intersection of the diagonals to non-parallel sides. For each property that will be proven for the trapezoid, a check shall be made with regards to the extent to which this property is retained in a quadrilateral that is not a trapezoid. At the same time, it is suggested and backed by a proof that some of the properties which hold true for the trapezoid also hold true for any convex quadrilateral, but their proof requires college-level knowledge. The paper illustrates the method of investigation "what if not", which permits guided investigations to be carried out.

2010 Mathematics Subject Classification: 51M04, 51M05.
Keywords and phrases: trapezoid, convex quadrilateral, harmonic quadruplet, the Appolonius circle, minimal perimeter, the complete quadrilateral, the homothety transformation, Menelaus' and Ceva's theorems.
Received February 24, 2016

Introduction

In the recent years, continued the process of discovery and characterization of special properties and development of formulas related to the trapezoid, convex quadrilateral (see [3], [5]).

The trapezoid is a particular case of a convex quadrilateral. Therefore, any property that holds in the quadrilateral also hold in the trapezoid. It is obvious that not any property that holds in the trapezoid will also hold in a convex quadrilateral.

For example, the following four properties, the three first properties are true for a trapezoid and are inherited by any convex quadrilateral (Properties A-C)[1], while the fourth property is only true for a trapezoid and is not inherited (Property D).

(A) A quadrilateral whose vertices are the middles of the sides of a trapezoid is a parallelogram. Varignon's theorem states that if we write "any quadrilateral" instead of a "trapezoid", the claim remains valid.

(B) In the trapezoid $ABCD\,(AB \| CD)$, it is given that E is the point of intersection of the diagonals, F is the concurrence point of the continuations of the sides AD and BC, EF intersects the sides AB and CD at the points G and H, respectively. Then there holds: $\frac{FG}{FH} = \frac{EG}{EH}\,(^*)$.

[1]Properties A and B are known properties, while C is a new property.

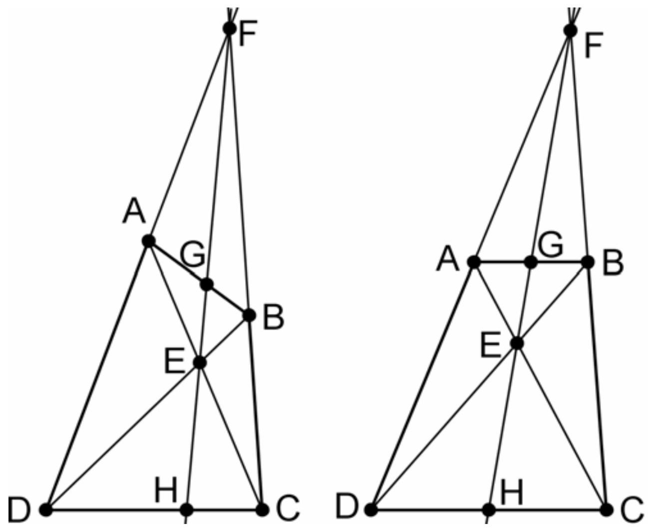

Figure 1.

Now, if in the formulation of the claim we write convex quadrilateral instead of trapezoid, the property $(^*)$ remains true.

(C) If in the trapezoid $ABCD\,(AB \| CD)$, we draw a circle through the point of intersection of the diagonals *E* and the point of intersection of the continuations of the legs *F*, so that the circle intersects the legs of the trapezoid at two interior points *M* and *N*, and the continuations of the diagonals at two additional points *K* and *L* (see Figure 2), then the straight lines *KN* and *LM* are concurrent at point *P* which lies on the side *AB*, and the straight lines *KM* and *LN* are concurrent at point Q which lies on the side *CD*.

Now, if in the formulation of the claim we write convex quadrilateral and a pair of opposite non-parallel sides instead of trapezoid and the legs of the trapezoid, the claim remains true (see [2]).

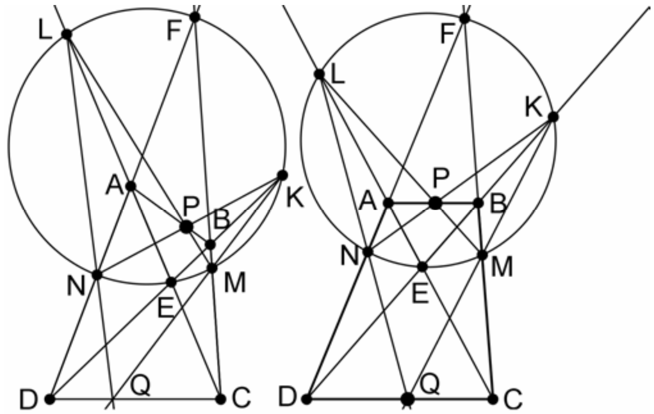

Figure 2.

(D) In the trapezoid $ABCD$ $(AB \| CD)$, EF is the midline, then there holds $EF = \dfrac{AB + CD}{2}$. In the quadrilateral $ABCD$ that is not a trapezoid, EF, the midline of the sides AD and BC (see Figure 3), then there holds $EF < \dfrac{AB + CD}{2}$.

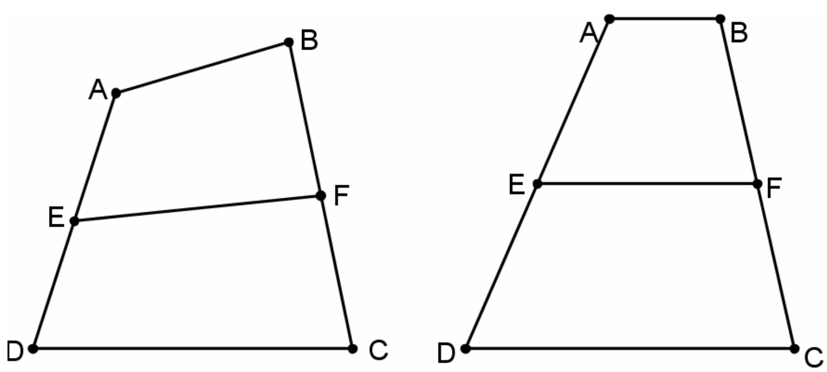

Figure 3.

This paper presents three properties that are common to a trapezoid and to a convex quadrilateral, which have to do with the drawing of perpendiculars from the point of concurrence of the diagonals to the sides (in a trapezoid) or to a pair of opposite sides (in a convex quadrilateral).

In all the properties, there is a difference between the formulation of the property for a trapezoid and for a quadrilateral, in the other words: in Properties 1 and 2, the difference is in the data of the property (the points I and J are defined in a different manner for a trapezoid and for an arbitrary quadrilateral), in Property 3, there is a difference in the requirement of the property. For Property 1 in the case of a trapezoid (Property 1.1) there are several proofs. In this paper, we present two of these proofs. The second proof is valid (with minor changes) for an arbitrary quadrilateral as well, and makes use of the concepts “harmonic quadruplet”, “the circle of Apollonius”, etc. The first proof uses only basic geometric concepts, such as: the midline in the triangle, the median to the hypotenuse in the right-angled triangle, properties of parallel angles, the property of angles in an isosceles triangle, proportions in the trapezoid, similarity of triangles etc.

1. The Property of a Perpendicular from the Point of Intersection of the Diagonals to one of the Opposite Non-parallel Sides

Property 1.1.

Given is an arbitrary trapezoid $ABCD$ $(AB \| CD)$, in which the point E is the point of intersection of the diagonals, the segment EM is perpendicular to the straight line BC $(M \in BC)$, the points I and J are the midpoints of the bases AB and CD, respectively (see Figure 4). The segment ME bisects the angle IMJ.

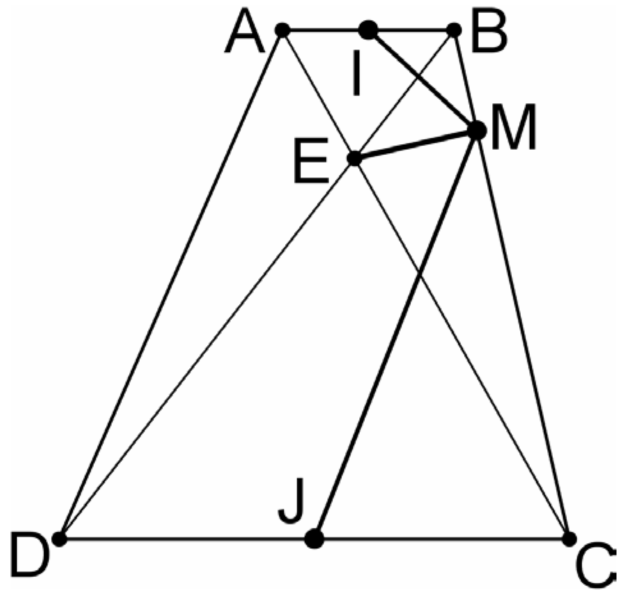

Figure 4.

First Proof.

We denote the middles of the segments EB and EC by T and U, respectively. We draw the segments IT, TM, MU, and UJ.

We denote: $\measuredangle BAC = \alpha$, $\measuredangle ABD = \beta$, $\measuredangle DBC = \gamma$, $\measuredangle ACB = \delta$ (see Figure 5).

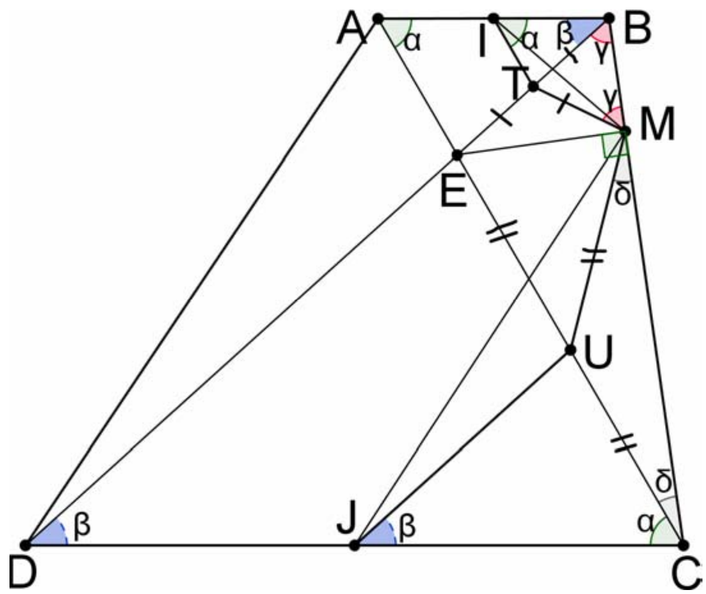

Figure 5.

Hence, from the properties of parallel angles, we obtain: $\measuredangle BIT = \alpha$, and $\measuredangle ACD = \alpha$, $\measuredangle BDC = \beta$, and $\measuredangle UJC = \beta$ (because IT is the midline in the triangle ABE and UJ is the midline in the triangle ECD). There also holds: $IT = \frac{1}{2} AE$, $UJ = \frac{1}{2} ED$.

We denote $AB = a$, $CD = b$. In the trapezoid $ABCD$, for the diagonal segments there holds: $\frac{AE}{EC} = \frac{BE}{ED} = \frac{a}{b}$.

In the right-angled triangles BME and MCE, the segments MT and MU are medians to the hypotenuses BE and EC, respectively. Therefore there holds: $MT = \frac{1}{2} BE$, $MU = \frac{1}{2} EC$.

When looking at the following ratios of segments: $\frac{IT}{MU} = \frac{\frac{1}{2}AE}{\frac{1}{2}EC} = \frac{AE}{EC}\left(= \frac{a}{b}\right)$ and $\frac{TM}{UJ} = \frac{\frac{1}{2}BE}{\frac{1}{2}ED} = \frac{BE}{ED}\left(= \frac{a}{b}\right)$, one finds that there holds: $\frac{IT}{MU} = \frac{TM}{UJ}\left(= \frac{a}{b}\right)(^{*})$.

By calculating the angles: $\measuredangle ABC + \measuredangle BCD = (\beta + \gamma) + (\delta + \alpha) = 180^{\circ}$, one obtains that: $\alpha + \beta + \gamma + \delta = 180^{\circ}$.

Using the following relation we calculate the angles $\measuredangle ITM$ and $\measuredangle MUJ$:

$$\measuredangle ITM = \measuredangle ITB + \measuredangle BTM = (180^{\circ} - \alpha - \beta) + (180^{\circ} - 2\gamma) = (\gamma + \delta) + (180^{\circ} - 2\gamma)$$

$$= 180^{\circ} - \gamma + \delta;$$

$$\measuredangle MUJ = \measuredangle MUE + \measuredangle EUJ = (\alpha + \beta) + 2\delta = (180^{\circ} - \gamma - \delta) + 2\delta$$

$$= 180^{\circ} - \gamma + \delta.$$

Therefore there holds: $\measuredangle ITM = \measuredangle MUJ(^{**})$.

From the formulas $(^{*})$ and $(^{**})$, it follows that the triangles *ITM* and *MUJ* are similar, therefore there holds $\frac{IM}{MJ} = \frac{a}{b}$. In the trapezoid *ABCD*, the points *I*, *E*, and *J* are located on the same straight line and there holds $\frac{IE}{EJ} = \frac{a}{b}$. Therefore, the following proportion holds: $\frac{IM}{MJ} = \frac{IE}{EJ}$. From this proportion, it follows that the segment *ME* is the bisector of the angle *IMJ* in the triangle *IMJ*. QED.

Second Proof.

We denote by *F* the point of intersection of the continuations of the sides *BC* and *AD* (see Figure 6).

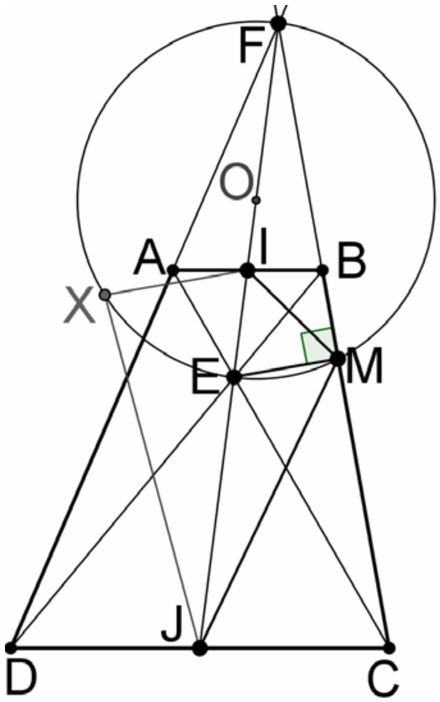

Figure 6.

It is well-known (see, for example, [8]) that for the four points E, F, I, and J, which are associated with the trapezoid as specified above, the following two claims hold:

(a) They lie on the same straight line.

(b) The following proportion of distances holds true: $\frac{IE}{EJ} = \frac{IF}{FJ}\left(= \frac{AB}{CD}\right)$.

From these properties, it follows that the points E and F belong to the locus of all the points the ratio of whose distances from the points I and J is a fixed value (that is equal to $\frac{AB}{CD}$). It is well-known that this locus is a circle whose diameter is the segment EF, in the other words, the circle of Apollonius for the segment IJ, which corresponds to the ratio $\frac{AB}{CD}$.

For any point X on the circle, there holds: $\frac{IX}{XJ} = \frac{AB}{CD} = \frac{IE}{EJ}$. In particular, for the point M on the circle, there holds $\frac{IM}{MJ} = \frac{IE}{EJ}$ (because $\triangle FIE = 90^{\circ}$). From the last proportion, it follows that the segment ME bisects the angle IMJ in the triangle IMJ. QED.

We shall now check if the property proven in 1.1 for a trapezoid is retained from a quadrilateral $ABCD$ that is not a trapezoid. In this case, the point I and J are not the midpoints of AB and CD. Therefore the theorem is formulated differently.

Property 1.2.

Let $ABCD$ be a convex quadrilateral, in which the point E is the point of intersection of the diagonals; F is the point of intersection of the sides BC and AD; I and J are points of intersection of the line EF with the sides AB and CD, respectively (see Figure 7). The segment EM is a perpendicular from the point E to the side BC. Then the segment ME bisects the angle IMJ.

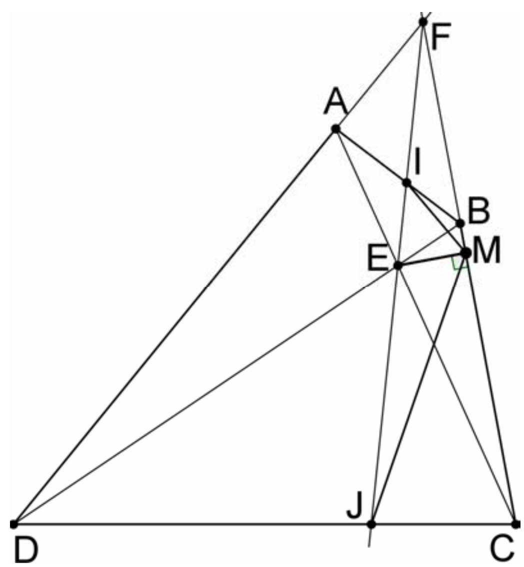

Figure 7.

Proof. We consider the complete quadrilateral *AFBECD*, in which *AB*, *EF*, and *CD* are diagonals. From a well-known property of the diagonals of a complete quadrilateral (see, for example, [4], Section 202), *each one of the diagonals of a complete quadrilateral is divided harmonically by the two other diagonals*.

In our case, the diagonal *EF* meets the diagonal *AB* at the point *I* (an interior point of the segment *EF*), and the diagonal *CD* at the point *J* (a point external to the segment *EF*). Based on this property, the four points *E*, *F*, *I*, and *J* form a harmonic quadruplet. In the other words, there holds $\frac{FI}{IE} = \frac{FJ}{JE}$ or $\frac{IF}{FJ} = \frac{IE}{EJ}\,(^*)$. From the equality $(^*)$, it follows that the circle whose diameter is the segment *EF* is a circle of Apollonius of the segment *IJ*, i.e., the locus of all the point the ratio of whose distances from the points *I* and *J* is a fixed value that equals $\frac{IE}{EJ}$. The point *M* belongs to this circle (because $\angle FME = 90°$), and therefore for *M* there also holds: $\frac{IM}{MJ} = \frac{IE}{EJ}$. Hence *ME* is the bisector of the angle $\angle IMJ$ in the triangle *IMJ*. QED.

Note. The property 1 also holds in the case when the point *M* belongs to the continuation of the side *BC* (see Figure 8).

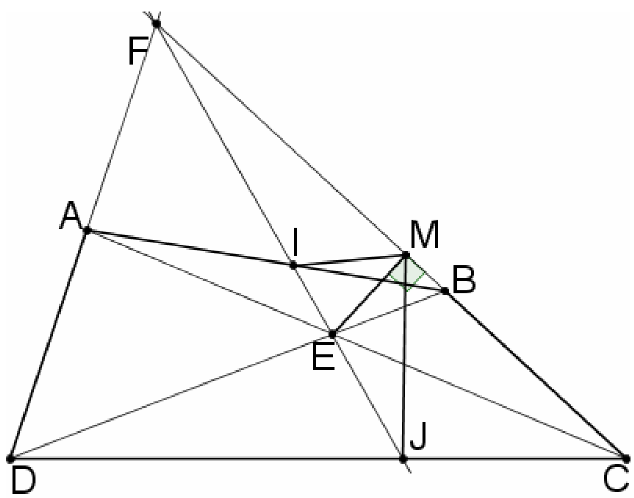

Figure 8.

2. Minimum Properties

Property 2.1.

Given is an arbitrary trapezoid $ABCD$ $(AB \| CD,\ BC \neq AD)$, in which the points E, M, I, and J are as described in the data of Property 1.1. In addition, it is given that the segment EN is perpendicular to the straight line $AD\,(N \in AD)$.

If the points M and N lie on the sides of the trapezoid (see Figure 9), then:

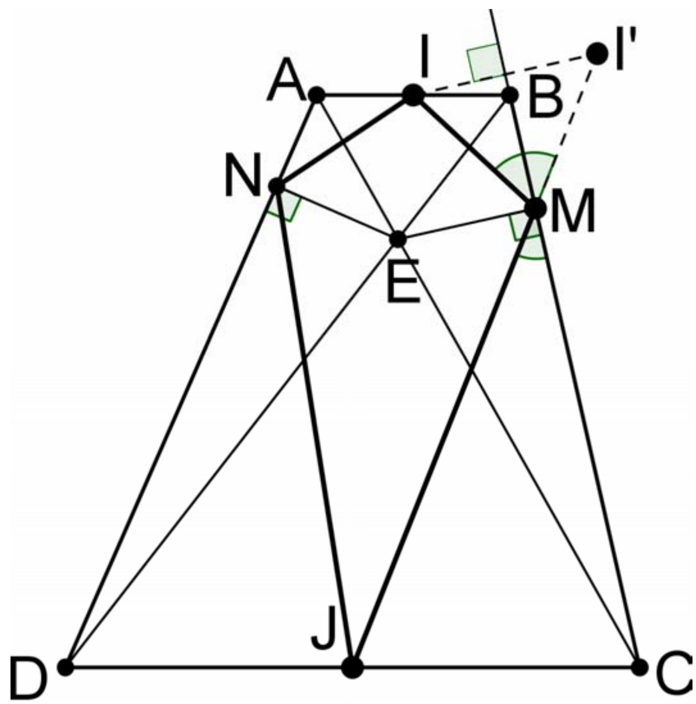

Figure 9.

(a) The quadrilateral $IMJN$ is the quadrilateral with the smallest perimeter among all the quadrilaterals $IXJY$ in which the points X and Y belong to the sides BC and AD, respectively.

(b) The opposite angle bisectors in the quadrilateral *IMJN* intersect on its diagonals (see Figure 10).

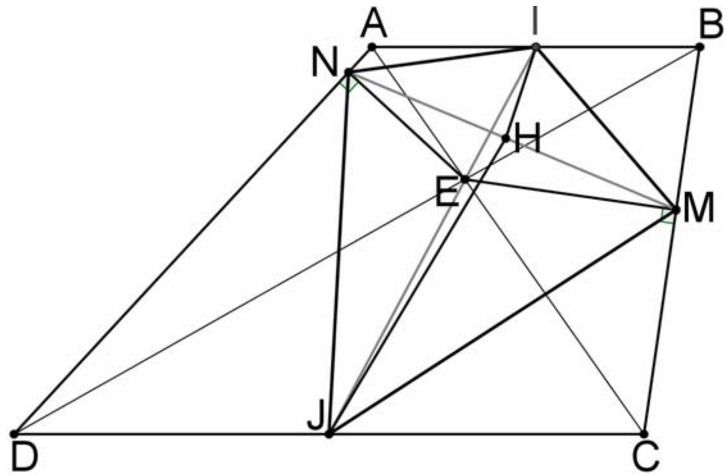

Figure 10.

(c) In the quadrilateral *IMJN* the products of the opposite side lengths are equal.

Notes. (1) If one of the points *M* or *N* lies on the continuation of the side, and the other is an interior point of the other side, then the quadrilateral *IMJA*, where *M* is an interior point of the side *BC* (see Figure 11), or the quadrilateral *IBJN*, where *N* is an interior point of the side *AD*, is the quadrilateral with the minimal perimeter.

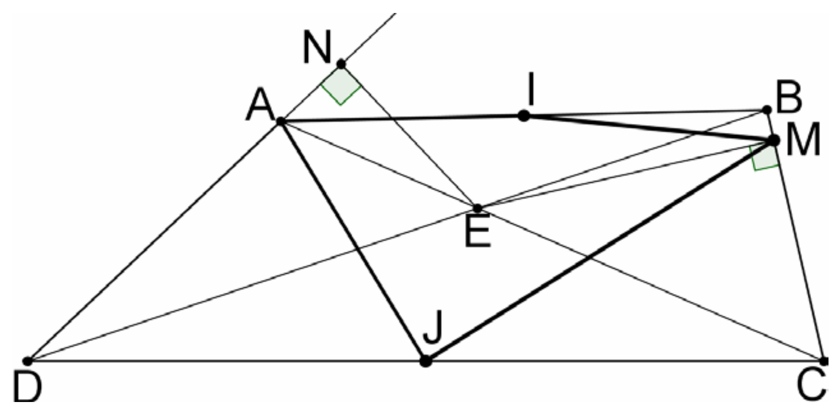

Figure 11.

(2) If the two points M and N lie on the continuations of the sides, then there does not exist a quadrilateral $IXJY$ that is inscribed in the trapezoid $ABCD$ and has a minimal perimeter.

Proof. (a) From the equality of the angles $\measuredangle EMB = \measuredangle EMC (= 90^{\circ})$, and $\measuredangle IME = \measuredangle EMJ$, we obtain the following equality of angles $\measuredangle IMB = \measuredangle JMC$. From the last equality, it is easy to conclude that the straight line JM passes through the point I', which is the point symmetrical to the point I relative to the straight line CB (see Figure 9). Therefore, the sum of the lengths of the segments $IM + MJ$ is equal to the length of the segment JI', or the length of the minimal path between the points I' and J. In a similar manner, the path $IN + NJ$ is the shortest of all the paths IYJ (where $Y \in AD$). Therefore, the perimeter of the quadrilateral $IMJN$ is the smallest.

(b) The angle bisectors $\measuredangle IMJ$ and $\measuredangle INJ$ of the quadrilateral $IMJN$ intersect at the point E of the diagonal IJ, and therefore, from the angle bisector theorem, we have: $\frac{IM}{MJ} = \frac{IE}{EJ} = \frac{IN}{NJ}$, or in another form: $\frac{IM}{IN} = \frac{JM}{JN} (^{*})$.

We denote by H the point of intersection of the angle bisector $\measuredangle NIM$ with the diagonal MN (see Figure 10). Therefore, from the angle bisector theorem, we have $\frac{MI}{IN} = \frac{MH}{HN} (^{**})$.

From the two proportions, it follows that $\frac{MJ}{JN} = \frac{MH}{HN}$, from which it follows that JH is the bisector of the angle $\measuredangle MJN$. In the other words, the angle bisectors intersect on the diagonal MN of the quadrilateral.

(c) From proportion $(^{*})$ follows another property of the quadrilateral $IMJN$: the product of the lengths of one pair of opposite sides equals the product of the lengths of the other pair of opposite sides. QED.

The second property is also true in any quadrilateral, but the definition of the points I and J is different.

Property 2.2.

Let $ABCD$ be a convex quadrilateral in which E is the point of intersection of the diagonals; F is the point of intersection of the continuations of the sides BC and AD; I and J are the points of intersection of the straight line EF with the sides AB and CD, respectively; the segments EM and EN are perpendiculars from the point E to the sides BC and AD, respectively. Then:

(a) The quadrilateral $IMJN$ is the quadrilateral with the minimal perimeter among all the quadrilaterals $IXJY$ in which the points X and Y belong to the sides BC and AD, respectively.

(b) The bisectors of the opposite angles in the quadrilateral $IMJN$ intersect on its diagonals (see Figure 12).

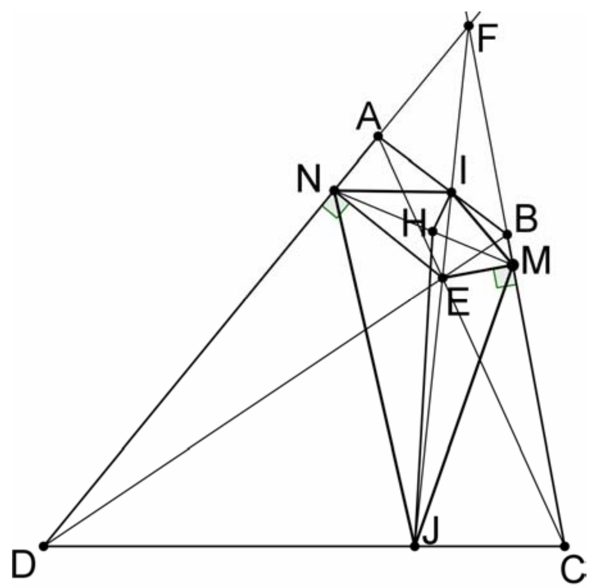

Figure 12.

(c) In the quadrilateral $IMJN$ the products of the opposite side lengths are equal.

Proof. The proof of Theorem 2.2 is identical to the proof of Theorem 2.1 (the proof of Theorem 2.1 does not make use of the fact that the points I and J bisect the sides AB and CD, rather only of the conclusion that ME is the bisector of angle IMJ).

Notes. (1) If, for example, the point N is an interior point of the side AD and the point M belongs to the continuation of the side BC (as described in Figure 13), then the quadrilateral $IBJY$ is the quadrilateral with the minimal perimeter among the quadrilaterals $IXJY$ that are inscribed in the quadrilateral $ABCD$.

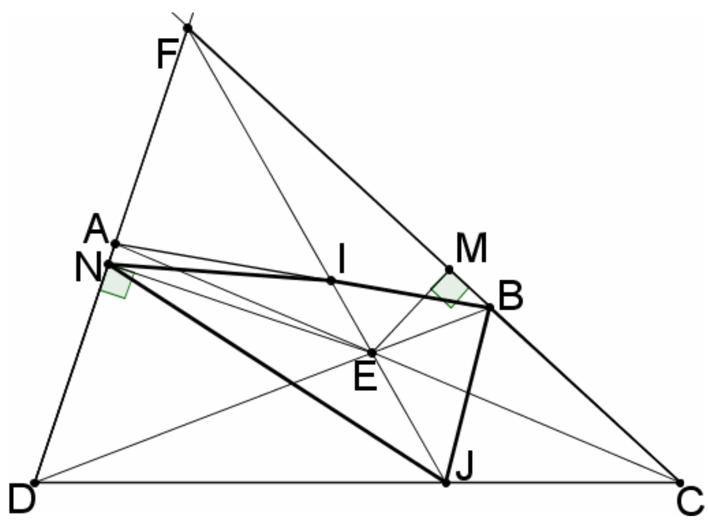

Figure 13.

(2) If the two points M and N lie on the continuations of the sides BC and AD, then there does not exist a quadrilateral $IXJY$ with a minimal perimeter that is inscribed in the quadrilateral $ABCD$.

3. Properties of the Continuations of the Sides *MN* and *KL* in the Quadrilateral *MNKL*

Property 3.1.

Given is an arbitrary trapezoid $ABCD$ ($AB \| CD$, $BC \neq AD$), in which E is the point of intersection of the diagonals; F is the point of intersection of the continuations of the sides BC and AD; MK is a straight line that passes through the point E, is perpendicular to the side BC at the point M, and intersects the side AD at the point K; NL is a straight line that passes through the point E, is perpendicular to the side AD at the point N, and intersects the side BC at the point L; T is the point of intersection of the straight lines KL and MN (see Figure 14).

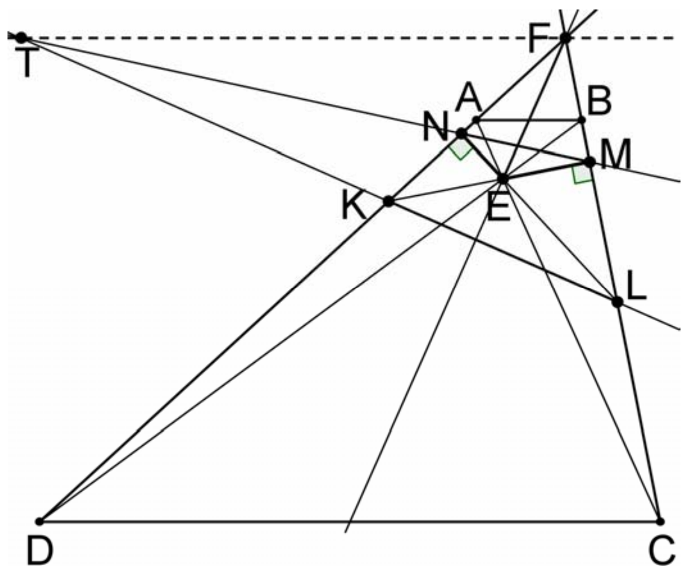

Figure 14.

Then the point T belongs to the straight line that passes through the point F, and is parallel to the straight line AB.

Proof. We make use of Menelaus' theorem with respect to the triangle FKL and the straight line MN that intersects the sides FL and FK of the triangle at the points M and N, respectively, and the continuation of the side KL at the point T.

According to this theorem, there holds: $\frac{MF}{ML} \cdot \frac{NK}{NF} \cdot \frac{TL}{TK} = 1\,(^*)$.

We denote by P the point of intersection of the straight line EF and the segment KL (see Figure 15). From Ceva's theorem for the three altitudes, in the triangle FKL there holds: $\frac{FN}{NK} \cdot \frac{KP}{PL} \cdot \frac{LM}{MF} = 1\,({}^{**})$.

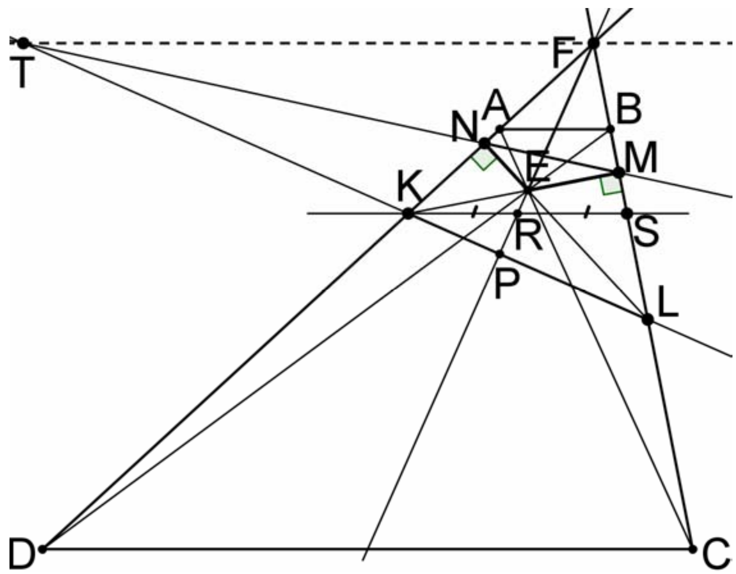

Figure 15.

We multiply the relations $({}^{*})$ and $({}^{**})$, and after canceling we obtain $\frac{TL}{TK} \cdot \frac{KP}{PL} = 1$, or $\frac{KT}{TL} = \frac{PL}{PK}\,({}^{***})$.

Through the point K, we draw the straight line $KS\,(S \in BC)$ that is parallel to the bases of the trapezoid AB and CD. The straight line EF bisects the segments KS (the point R in Figure 15) because it bisects the bases of the trapezoid.

From Menelaus' theorem for the triangle KLS and the straight line EF that intersects the sides KL and SK of the triangle at the points P and R, respectively, and the continuation of the side LS at the point F, we have: $\frac{FL}{FS} \cdot \frac{RS}{RK} \cdot \frac{PK}{PL} = 1$, and hence, since $\frac{RS}{RK} = 1$, we obtain the proportion $\frac{FL}{FS} = \frac{PL}{PK}$, and from it and the proportion $({}^{***})$ we obtain $\frac{FL}{FS} = \frac{KT}{TL}$ or $\frac{LS}{SF} = \frac{LK}{KT}$.

From the last equality and from the converse of Thales' theorem, it follows that $KS \| FT$, and therefore also $FT \| AB$. QED.

Property 3.2.

Let *ABCD* be a convex quadrilateral in which: *E* is the point of intersection of the diagonals; *F* is the point of intersection of the continuations of the sides *BC* and *AD*; *G* is the point of intersection of the continuations of the sides *AB* and *CD*; *MK* is a straight line that passes through the point *E*, is perpendicular to the side *BC* at the point *M*, and intersects the side *AD* at the point *K*; *NL* is a straight line that passes through the point *E*, is perpendicular to the side *AD* at the point *N*, and intersects the side *BC* at the point *L*; *T* is the point of intersection of the straight lines *KL* and *MN* (see Figure 16). Then the point *T* belongs to the straight line *FG*.

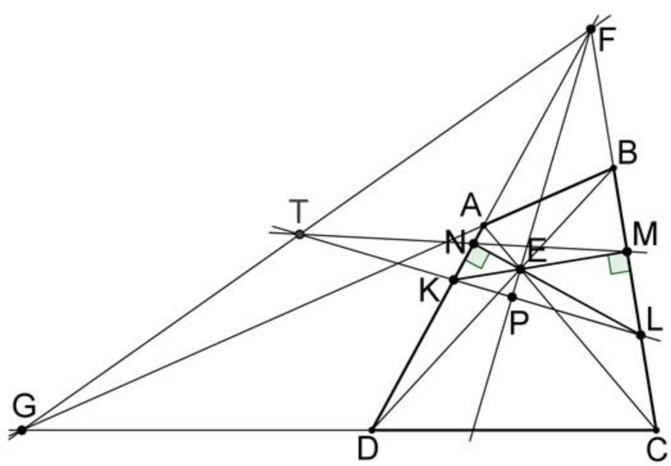

Figure 16.

Proof. Through the point G, we draw a straight line that is perpendicular to the line EF, and denote by U, V, and W the points of its intersection with the straight lines EF, AD, and BC, respectively (Figure 17).

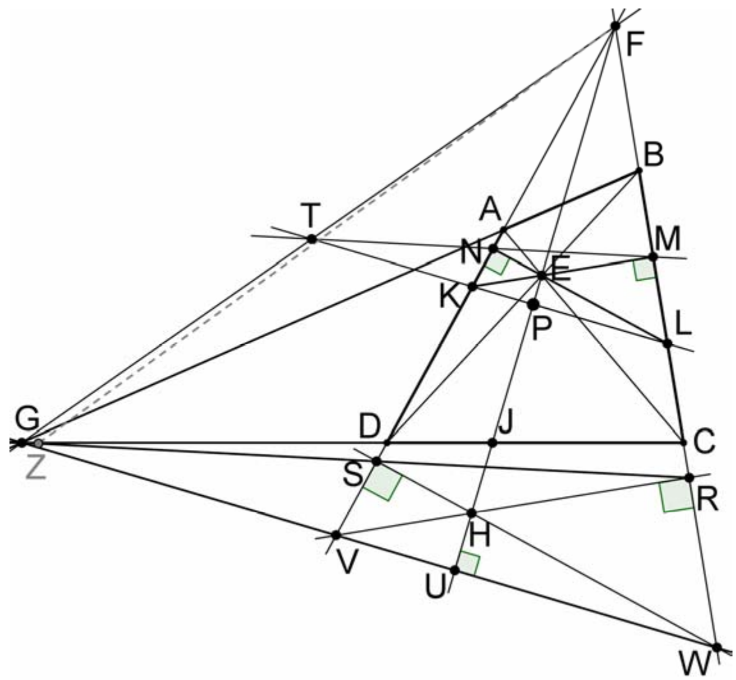

Figure 17.

In the triangle FVW, we draw the altitudes VR and WS, and denote by H the point of their intersection. The third altitude FU also passes through the point H. We draw a straight line RS and denote by Z the point of its intersection with the straight line GW. We shall prove that the points G and Z coincide.

The four points C, D, J, and G on the straight line CD form a harmonic quadruplet (which follows from the properties of the diagonals of the complete quadrilateral given in proof of Property 1.2, which states that in the complete quadrilateral $AFBECD$ the diagonal CD is divided by the diagonals EF and AB by a harmonic division).

Therefore, the four rays FC, FD, FJ, and FG form a harmonic quadruplet.

The four points W, V, U, and Z on the straight line GW also form a harmonic quadruplet (as division of the diagonal VW of the complete quadrilateral $SFRHWV$ by the other diagonals FH and RS).

Therefore, the four rays FW, FV, FU, and FZ form a harmonic quadruplet.

We obtained two harmonic quadruplets, in which the three rays of one quadruplet coincide with the three rays of the other quadruplet. Therefore, the last rays on the quadruplets, rays FG and FZ, also coincide with each other. Since the points G and Z of the rays lie on the same straight line WV, the points G and Z coincide.

Hence it follows that the straight lines RS and WV intersect at the point G.

Now let us consider the homothetic transformation with the center at the point F and the similarity factor $k = \dfrac{FP}{FU}$. In this transformation, the straight line VW shall be transformed into the straight line KL, the straight line VR – to the perpendicular KM, the perpendicular WS – to the perpendicular LN, the straight line RS – to the straight line MN. Therefore, the point G which is the point of intersection of the straight lines RS and WV will transform into T, which is the point of intersection of the straight lines MN and LK, and therefore the point T belongs to the straight line FG. QED.

Conclusion from Theorem 3.2 (General proof of Theorem 3.1)**.**

We employ concepts from projective geometry, where all parallel lines intersect at a certain point at infinity, and denote by G the infinity point at which the bases of the trapezoid AB and CD intersect, all the lines parallel to the line AB pass through this point.

Therefore, the line *FG* is parallel to *AB* and passes through the point *F*. Therefore, based on Property 3.2, the point *T*, which is the point of intersection of the continuations of the sides *KL* and *MN* of the quadrilateral *KLMN* belongs to the straight line *FG* that is parallel to the straight line *AB*. QED.

Summary

This paper illustrates the possibilities that open before the researcher who asks "does the proven theorem have a generalization?". Some very interesting properties of trapezoids that were discovered in this paper led to the formulation of hypotheses concerning an arbitrary quadrilateral. These hypotheses were checked using *GeoGebra* software, and were later proven as true. Therefore, the properties found for the trapezoid were generalized for an arbitrary convex quadrilateral.

In addition, in the process of searching for new properties that are common both for a trapezoid and an arbitrary quadrilateral that is carried out during the research it was found that it is usually easier to discover such properties in the case of the trapezoid. The process has shown that if we find it difficult to find a proof that is based on the properties of the trapezoid, it is worthwhile to search for a more general proof for an arbitrary quadrilateral. For example, this was illustrated in the proofs of Properties 3 and in the proof of example (C) that appears in the introduction (see [2]).

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