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APPLICATION OF ADJACENCY AND GENERALIZED INVERSE MATRICES TO ORDERING THE THREE OPTIMAL $(4 \times 4)/4$ SEMI-LATIN SQUARES

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Abstract

In this work, we present the adjacency and generalized inverse matrices of the three optimal $(4 \times 4)/4$ semi-Latin squares. These matrices are then used in conjunction with each other to discriminate amongst the squares by computing and comparing the variance of adjacency induced by the treatments in each square with those of the other squares, with the aid of MATLAB. The square which minimizes both the maximum variance of adjacency and the number of distinct values of the variance of adjacency amongst the squares, and where tie occurs, also have a minimum value of this variance is considered most

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preferable, and so on. Results show that the square, Ω_2 is the most preferable for experimentation, while Ω_1 is preferable to Ω_3, which is consistent with earlier results by Uto and Chigbu [23].

1. Introduction

Designs for experiments are generally assessed for purposes of preference on the basis of some well-defined criteria known as optimality criteria, which follows from the corresponding efficiency factor. Some of the popular optimality criteria include the *A*-, *D*- and *E*-criteria, each of them being a function of the canonical efficiency factors of the design. Given a class of designs Ω, any design $\omega \in \Omega$ which maximizes the value of the efficiency factor among all the designs in Ω is said to be optimal in Ω; see Bailey and Royle [5].

A $(4 \times 4)/4$ semi-Latin square is a row-column design which has four rows and four columns with four treatments per row-column intersection. There are sixteen treatments in the design which are allocated to the plots in a manner that each treatment appears once in each row and once in each column; see, for example, Bailey and Chigbu [4], Bailey [2, 3], as well as Bailey and Royle [5]. Semi-Latin squares have been found useful for experiments in diverse fields; ranging from agriculture and industry to consumer testing; see Preece and Freeman [18], as well as Bailey [2, 3] for some of its uses.

The semi-Latin squares considered in this work, $\Omega_1, \Omega_2,$ and Ω_3 which are given in Figures 1, 2, and 3, respectively, have been found to be *A*-, *D*- and *E*-optimal by Chigbu [8]. Though equally optimal with respect to the aforementioned criteria, these designs have different treatment concurrences which suggests some inherent differences existing among them; see Chigbu [8, 9]. Chigbu [9] found the best, the most preferable, of these squares using an analytic approach by obtaining the variance of elementary contrasts of treatments for each square and comparing the resulting contrasts with minimum variances among the squares; but this approach did not induce a proper ordering of the squares. Sequel to this, Chigbu [10] adopted a numerical approach which involves computation of generalized inverses of the information matrices of these squares and thus ascertained the same square due to Chigbu [9] as the best and rated the other two the same on a common basis. Uto and Chigbu [23] distinguished each of these squares from another using a near regular graph design approach by computing the variance of the difference in concurrences between pairs of treatments for each square and comparing the results for minimum variance among the squares. Their best design corresponds to the one of Chigbu [9, 10].

In this paper, we set out to further distinguish among these squares with a view to ordering them to know which should be preferred to the other for experimentation using a different approach, which involves computing the variance of adjacency for each treatment in each of these designs and then comparing the results for each design with those of the other competing designs.

2. Preliminaries

A semi-Latin square has its treatments orthogonal to both the rows and columns; by ignoring the rows and columns, the resulting design is a doubly-resolvable incomplete-block design, which is its quotient block design; see Bailey [2]. Hence, a semi-Latin square is usually assessed for efficiency as a binary incomplete-block design where each row-column intersection is regarded as a block of its equivalent incomplete-block design, and symbols its treatments. There are n^2 blocks each of size k, and nk treatments each occurring n times in the design; see Bailey [2, 3] as well as Bailey and Royle [5].

Incomplete block designs are usually, associated with four matrices: the incidence matrix denoted by N, whose (i, j)-th entries are the number of times treatment i appears in block j; the concurrence matrix

NN'; the information matrix $L = rI - \frac{1}{k} NN'$ (r being the number of replications of each treatment in the design = n, the number of rows or columns of a semi-Latin square; I a conformable identity matrix); and the variance-covariance matrix Q. The variance-covariance matrix is any generalized-inverse of L. The concurrence matrix is important in developing the theory of incomplete block designs. Generally, while the concurrence matrix summarizes the design, the variance-covariance matrix summarizes the analysis; see Chigbu [7].

The adjacency matrix is like the concurrence matrix except that the leading diagonal entries are all zeros. The information matrix L is a sub-rank matrix of order t $(\text{rank}(L) \leq t - 1)$, t being the number of treatments in the design $= nk$ for this work. In particular, $\text{rank}(L) = t - 1$ for connected designs, but for disconnected designs $\text{rank}(L) < t - 1$; see, for example, Onukogu and Chigbu [14], Chigbu [7], as well as Cameron *et al.* [6]. The L matrix is thus singular and non-invertible; hence, its generalized inverse is usually sought.

The adjacency and generalized matrices have found several applications. For instance, while the adjacency matrix (as well as its powers) plays a prominent role in graph theory, where it is used to determine the number of circuits as well as the number of distinct paths of certain lengths from the i-th to the j-th vertex of its variety-concurrence graph; see, for example, Paterson [15], Wild [24], Raghavarao [19], as well as Onukogu and Chigbu [14]; the generalized inverse matrix plays an important role in linear algebra in determining the solutions of linear equations when the coefficient matrix has no inverse; see, for example, Searle [21], Penrose [16] and Greville [12]. Moreover, the generalized inverse matrix is used to find the variance of contrasts between pairs of treatments for connected designs; see, for example, Cameron *et al.* [6], Chigbu [7, 9, 10] as well as Onukogu and Chigbu [14]. The most widely known type of generalized inverse matrix is the Moore-Penrose generalized inverse.

A α a 1	B β b 2	C γ c 3	D δ d 4
B γ d 2	A δ c 1	D α b 4	C β a 3
C δ b 4	D γ a 3	A β d 2	B α c 1
D β c 3	C α d 4	B δ a 1	A γ b 2

Figure 1. The semi-Latin square, Ω_1.

A_1 A_2 α a	B_1 B_2 β b	C_1 C_2 γ c	D_1 D_2 δ d
B_1 B_2 γ d	A_1 A_2 δ c	D_1 D_2 α b	C_1 C_2 β a
C_1 C_2 δ b	D_1 D_2 γ a	A_1 A_2 β d	B_1 B_2 α c
D_1 D_2 β c	C_1 C_2 α d	B_1 B_2 δ a	A_1 A_2 γ b

Figure 2. The semi-Latin square, Ω_2.

A α a 1	B β b 2	C γ c 3	D δ d 4
B γ d 2	A δ c 1	D α b 4	C β a 3
C δ b 4	D γ a 3	A β d 1	B α c 2
D β c 3	C α d 4	B δ a 2	A γ b 1

Figure 3. The semi-Latin square, Ω_3.

3. Definitions

Definition 3.1 (Connected design)**.** A connected design is one in which all elementary contrasts of treatments are estimable. Equivalently, a design is said to be connected if, given any two treatments α and β, it is possible to construct a chain of treatments $\alpha = \alpha_0, \alpha_1, \alpha_2, \ldots, \alpha_n = \beta$ such that every consecutive pair of treatments in the chain occurs together in a block.

Definition 3.2 (Variety-concurrence graph)**.** The variety-concurrence graph, $G(\Im)$, of a block design $\Im$ is a graph with treatments as vertices and the number of edges between any two treatments τ_i and τ_j $(i \neq j)$ equals the number of blocks which τ_i and τ_j occur together in $\Im$.

Definition 3.3 (Adjacency matrix)**.** The adjacency matrix of the variety-concurrence graph of an incomplete-block design is the square matrix, A, of order t, whose (i, j)-th entry $(i \neq j)$ is the number of lines joining the points i and j, taken to be zero if $i = j$. It is given by

$$A = NN' - rI, \tag{3.3.1}$$

where N is the treatment-by-block incidence matrix of order $t \times b$ (t and b denoting the number of treatments and blocks, respectively, in the design); NN' the $t \times t$ concurrence matrix; r the number of replications of each treatment in the design = n in this work; and I a conformable identity matrix.

Definition 3.4 (Moore-Penrose generalized inverse)**.** Given a matrix $A \in M_{m,n}$, the Moore-Penrose generalized inverse of A, denoted A^+, is the unique matrix in $M_{n,m}$ satisfying the conditions:

(1) $AA^+A = A$,

(2) $A^+AA^+ = A^+$,

(3) $(AA^+)' = AA^+$,

(4) $(A^+A)' = A^+A$;

see, for example, Penrose [16], Plemmons and Cline [17], Searle [21], Fill and Fishkind [11], as well as Rakha [20].

4. Methods

The adjacency matrix, $A_i = N_i N_i' - nI;\ i = 1, 2, 3$ of the variety-concurrence graph for the semi-Latin squares $\Omega_i (i = 1, 2, 3)$, respectively, under consideration were adapted from Uto and Chigbu [22], which were generated with the aid of MATLAB. Subsequently, the generalized inverse L_i^+, $i = 1, 2, 3$ of the respective information matrix $L_i = nI - k^{-1} N_i N_i';\ i = 1, 2, 3$ was generated also via MATLAB.

We note that for connected designs, the matrix $(L + aJ)$, a being a scalar multiple and J an all-ones matrix is non-singular and invertible for any $a \neq 0$ and its inverse

$$L^+ = (L + aJ)^{-1}, \tag{4.1}$$

is a generalized inverse of L, the information matrix; see, for example, Cameron *et al.* [6] and Chigbu [10]. Each generalized inverse L_i^+, $i = 1, 2, 3$ in this work was obtained by setting $a = 1$ in Equation (4.1). Hence

$$L_i^+ = (L_i + J)^{-1}, \quad \forall i = 1, 2, 3. \tag{4.2}$$

Each generalized inverse L_i^+ satisfies the Moore-Penrose inverse properties given in the preceding section with respect to the L_i and even the $(L + J)$ matrices. In some algebraic sense, the all-ones matrix, J, in conjunction with an identity matrix, I, of the same order span some subspace of the real vector space associated with each design. The all-ones matrix is analogous to the sum of all the zero-one matrices of order sixteen that make up the association scheme on the set of sixteen treatments of each of the semi-Latin squares; see Chigbu [10] and Cameron *et al.* [6].

Again, we note that design optimality criteria are real-valued functions of the matrix L^{+} that it is desirable to minimize; see Cameron *et al*. [6]. Moreover, we remind that a G-optimal design minimizes the maximum variance of the estimate of a surface (response), given by $\min[\max(\underline{x}'M^{-1}(\xi)\underline{x})]$, where $\underline{x}'$ is a p-component row vector with components corresponding to the entries of each row of the design matrix, X; and $M^{-1}(\xi)$, the inverse of the information matrix; see, for example, Onukogu [13], Atkinson and Donev [1] as well as Onukogu and Chigbu [14].

Furthermore, as given by Cameron *et al*. [6], a design is optimal if the number of its distinct pairwise treatment variances is fewest when compared with those of the others in the same class with it.

We then computed for each design, using MATLAB, the variance of adjacency induced by each treatment, j in the design, given by

$$\operatorname{Var}(A_j) = \underline{u}'_j L^{+} \underline{u}_j, \quad j = 1, 2, \ldots, nk, \tag{4.3}$$

where $\underline{u}'_j$ denotes an nk-component row vector with components the entries of the j-th row of the adjacency matrix, A, of the variety-concurrence graph of the design; and L^{+}, the generalized inverse of the information matrix of the design, as given in Equation (4.2). The results of the computed variances were then compared with each other amongst the squares. The square which satisfies the property: $\min[\max(\underline{u}'_j L^{+} \underline{u}_j)]$ over all the squares, and also has a minimum number of distinct values of this variance amongst them becomes the most preferable. In particular, if tie occurs, the square having a minimum variance is the most preferable, and so on.

5. Results and Discussion

The adjacency matrices A_1, A_2, and A_3 of the variety-concurrence graph of the respective semi-Latin squares Ω_1, Ω_2, and Ω_3 are presented in Tables 1, 2, and 3.

Table 1. The incidence matrix, A_1 for Ω_1

$$A_1 = \begin{matrix}
0 & 1 & 1 & 2 & 0 & 1 & 1 & 2 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \\
1 & 0 & 1 & 2 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 2 \\
1 & 1 & 0 & 2 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 2 & 1 & 1 & 0 & 0 \\
2 & 2 & 2 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 2 & 0 & 0 \\
0 & 1 & 1 & 2 & 0 & 1 & 1 & 2 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \\
1 & 0 & 1 & 0 & 1 & 0 & 1 & 2 & 1 & 0 & 1 & 2 & 1 & 0 & 1 & 0 \\
1 & 1 & 0 & 0 & 1 & 1 & 0 & 2 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 2 \\
2 & 0 & 0 & 0 & 2 & 2 & 2 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 2 & 0 \\
0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 2 & 0 & 1 & 1 & 2 \\
1 & 0 & 1 & 0 & 1 & 0 & 1 & 2 & 1 & 0 & 1 & 2 & 1 & 0 & 1 & 0 \\
1 & 1 & 0 & 2 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 2 & 1 & 1 & 0 & 0 \\
0 & 0 & 2 & 0 & 0 & 2 & 0 & 0 & 2 & 2 & 2 & 0 & 2 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 2 & 0 & 1 & 1 & 2 \\
1 & 0 & 1 & 2 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 2 \\
1 & 1 & 0 & 0 & 1 & 1 & 0 & 2 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 2 \\
0 & 2 & 0 & 0 & 0 & 0 & 2 & 0 & 2 & 0 & 0 & 0 & 2 & 2 & 2 & 0
\end{matrix}$$

Table 2. The incidence matrix, A_2 for Ω_2

$$A_2 = \begin{matrix}
0 & 4 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\
4 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\
1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 \\
1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 \\
0 & 0 & 1 & 1 & 0 & 4 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\
0 & 0 & 1 & 1 & 4 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\
1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 \\
1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 4 & 1 & 1 & 0 & 0 & 1 & 1 \\
0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 4 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\
1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 \\
1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 4 & 1 & 1 \\
0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 4 & 0 & 1 & 1 \\
1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 \\
1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0
\end{matrix}$$

Table 3. The incidence matrix, A_3 for Ω_3

$$A_3 = \begin{matrix}
0 & 1 & 1 & 4 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \\
1 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 2 \\
1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 2 & 1 & 1 & 0 & 0 \\
4 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \\
0 & 1 & 1 & 0 & 0 & 1 & 1 & 4 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \\
1 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 2 & 1 & 0 & 1 & 0 \\
1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 2 \\
0 & 1 & 1 & 0 & 4 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \\
0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 2 & 0 & 1 & 1 & 2 \\
1 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 2 & 1 & 0 & 1 & 0 \\
1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 2 & 1 & 1 & 0 & 0 \\
0 & 0 & 2 & 0 & 0 & 2 & 0 & 0 & 2 & 2 & 2 & 0 & 2 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 2 & 0 & 1 & 1 & 2 \\
1 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 2 \\
1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 2 \\
0 & 2 & 0 & 0 & 0 & 0 & 2 & 0 & 2 & 0 & 0 & 0 & 2 & 2 & 2 & 0
\end{matrix}$$

The generalized inverse matrices L_1^+, L_2^+, and L_3^+ of the information matrix of these squares are given in Tables 4, 5, and 6, respectively.

Table 4. The generalized inverse matrix L_1^+ of the information matrix of the square Ω_1

$$L_1^+ = \begin{matrix}
0.3112 & -0.0117 & -0.0117 & 0.0195 & -0.0221 & -0.0117 & -0.0117 & 0.0195 & -0.0430 & -0.0117 & -0.0117 & -0.0430 & -0.0430 & -0.0117 & -0.0117 & -0.0430 \\
-0.0117 & 0.3112 & -0.0117 & 0.0195 & -0.0117 & -0.0430 & -0.0117 & -0.0430 & -0.0117 & -0.0430 & -0.0117 & -0.0430 & -0.0117 & -0.0221 & -0.0117 & 0.0195 \\
-0.0117 & -0.0117 & 0.3112 & 0.0195 & -0.0117 & -0.0117 & -0.0430 & -0.0430 & -0.0117 & -0.0117 & -0.0221 & 0.0195 & -0.0117 & -0.0117 & -0.0430 & -0.0430 \\
0.0195 & 0.0195 & 0.0195 & 0.3320 & 0.0195 & -0.0430 & -0.0430 & -0.0430 & -0.0430 & -0.0430 & 0.0195 & -0.0430 & -0.0430 & 0.0195 & -0.0430 & -0.0430 \\
-0.0221 & -0.0117 & -0.0117 & 0.0195 & 0.3112 & -0.0117 & -0.0117 & 0.0195 & -0.0430 & -0.0117 & -0.0117 & -0.0430 & -0.0430 & -0.0117 & -0.0117 & -0.0430 \\
-0.0117 & -0.0430 & -0.0117 & -0.0430 & -0.0117 & 0.3112 & -0.0117 & 0.0195 & -0.0117 & -0.0221 & -0.0117 & 0.0195 & -0.0117 & -0.0430 & -0.0117 & -0.0430 \\
-0.0117 & -0.0117 & -0.0430 & -0.0430 & -0.0117 & -0.0117 & 0.3112 & 0.0195 & -0.0117 & -0.0117 & -0.0430 & -0.0430 & -0.0117 & -0.0117 & -0.0221 & 0.0195 \\
0.0195 & -0.0430 & -0.0430 & -0.0430 & 0.0195 & 0.0195 & 0.0195 & 0.3320 & -0.0430 & 0.0195 & -0.0430 & -0.0430 & -0.0430 & -0.0430 & 0.0195 & -0.0430 \\
-0.0430 & -0.0117 & -0.0117 & -0.0430 & -0.0430 & -0.0117 & -0.0117 & -0.0430 & 0.3112 & -0.0117 & -0.0117 & 0.0195 & -0.0221 & -0.0117 & -0.0117 & 0.0195 \\
-0.0117 & -0.0430 & -0.0117 & -0.0430 & -0.0117 & -0.0221 & -0.0117 & 0.0195 & -0.0117 & 0.3112 & -0.0117 & 0.0195 & -0.0117 & -0.0430 & -0.0117 & -0.0430 \\
-0.0117 & -0.0117 & -0.0221 & 0.0195 & -0.0117 & -0.0117 & -0.0430 & -0.0430 & -0.0117 & -0.0117 & 0.3112 & 0.0195 & -0.0117 & -0.0117 & -0.0430 & -0.0430 \\
-0.0430 & -0.0430 & 0.0195 & -0.0430 & -0.0430 & 0.0195 & -0.0430 & -0.0430 & 0.0195 & 0.0195 & 0.0195 & 0.3320 & 0.0195 & -0.0430 & -0.0430 & -0.0430 \\
-0.0430 & -0.0117 & -0.0117 & -0.0430 & -0.0430 & -0.0117 & -0.0117 & -0.0430 & -0.0221 & -0.0117 & -0.0117 & 0.0195 & 0.3112 & -0.0117 & -0.0117 & 0.0195 \\
-0.0117 & -0.0221 & -0.0117 & 0.0195 & -0.0117 & -0.0430 & -0.0117 & -0.0430 & -0.0117 & -0.0430 & -0.0117 & -0.0430 & -0.0117 & 0.3112 & -0.0117 & 0.0195 \\
-0.0117 & -0.0117 & -0.0430 & -0.0430 & -0.0117 & -0.0117 & -0.0221 & 0.0195 & -0.0117 & -0.0117 & -0.0430 & -0.0430 & -0.0117 & -0.0117 & 0.3112 & 0.0195 \\
-0.0430 & 0.0195 & -0.0430 & -0.0430 & -0.0430 & -0.0430 & 0.0195 & -0.0430 & 0.0195 & -0.0430 & -0.0430 & -0.0430 & 0.0195 & 0.0195 & 0.0195 & 0.3320
\end{matrix}$$

Table 5. The generalized inverse matrix L_2^+ of the information matrix of the square Ω_2

$$
L_2^+ = \begin{bmatrix}
0.3320 & 0.0820 & -0.0117 & -0.0117 & -0.0430 & -0.0430 & -0.0117 & -0.0117 & -0.0430 & -0.0430 & -0.0117 & -0.0117 & -0.0430 & -0.0430 & -0.0117 & -0.0117 \\
0.0820 & 0.3320 & -0.0117 & -0.0117 & -0.0430 & -0.0430 & -0.0117 & -0.0117 & -0.0430 & -0.0430 & -0.0117 & -0.0117 & -0.0430 & -0.0430 & -0.0117 & -0.0117 \\
-0.0117 & -0.0117 & 0.3008 & -0.0117 & -0.0117 & -0.0117 & -0.0326 & -0.0117 & -0.0117 & -0.0117 & -0.0326 & -0.0117 & -0.0117 & -0.0117 & -0.0326 & -0.0117 \\
-0.0117 & -0.0117 & -0.0117 & 0.3008 & -0.0117 & -0.0117 & -0.0117 & -0.0326 & -0.0117 & -0.0117 & -0.0117 & -0.0326 & -0.0117 & -0.0117 & -0.0117 & -0.0326 \\
-0.0430 & -0.0430 & -0.0117 & -0.0117 & 0.3320 & 0.0820 & -0.0117 & -0.0117 & -0.0430 & -0.0430 & -0.0117 & -0.0117 & -0.0430 & -0.0430 & -0.0117 & -0.0117 \\
-0.0430 & -0.0430 & -0.0117 & -0.0117 & 0.0820 & 0.3320 & -0.0117 & -0.0117 & -0.0430 & -0.0430 & -0.0117 & -0.0117 & -0.0430 & -0.0430 & -0.0117 & -0.0117 \\
-0.0117 & -0.0117 & -0.0326 & -0.0117 & -0.0117 & -0.0117 & 0.3008 & -0.0117 & -0.0117 & -0.0117 & -0.0326 & -0.0117 & -0.0117 & -0.0117 & -0.0326 & -0.0117 \\
-0.0117 & -0.0117 & -0.0117 & -0.0326 & -0.0117 & -0.0117 & -0.0117 & 0.3008 & -0.0117 & -0.0117 & -0.0117 & -0.0326 & -0.0117 & -0.0117 & -0.0117 & -0.0326 \\
-0.0430 & -0.0430 & -0.0117 & -0.0117 & -0.0430 & -0.0430 & -0.0117 & -0.0117 & 0.3320 & 0.0820 & -0.0117 & -0.0117 & -0.0430 & -0.0430 & -0.0117 & -0.0117 \\
-0.0430 & -0.0430 & -0.0117 & -0.0117 & -0.0430 & -0.0430 & -0.0117 & -0.0117 & 0.0820 & 0.3320 & -0.0117 & -0.0117 & -0.0430 & -0.0430 & -0.0117 & -0.0117 \\
-0.0117 & -0.0117 & -0.0326 & -0.0117 & -0.0117 & -0.0117 & -0.0326 & -0.0117 & -0.0117 & -0.0117 & 0.3008 & -0.0117 & -0.0117 & -0.0117 & -0.0326 & -0.0117 \\
-0.0117 & -0.0117 & -0.0117 & -0.0326 & -0.0117 & -0.0117 & -0.0117 & -0.0326 & -0.0117 & -0.0117 & -0.0117 & 0.3008 & -0.0117 & -0.0117 & -0.0117 & -0.0326 \\
-0.0430 & -0.0430 & -0.0117 & -0.0117 & -0.0430 & -0.0430 & -0.0117 & -0.0117 & -0.0430 & -0.0430 & -0.0117 & -0.0117 & 0.3320 & 0.0820 & -0.0117 & -0.0117 \\
-0.0430 & -0.0430 & -0.0117 & -0.0117 & -0.0430 & -0.0430 & -0.0117 & -0.0117 & -0.0430 & -0.0430 & -0.0117 & -0.0117 & 0.0820 & 0.3320 & -0.0117 & -0.0117 \\
-0.0117 & -0.0117 & -0.0326 & -0.0117 & -0.0117 & -0.0117 & -0.0326 & -0.0117 & -0.0117 & -0.0117 & -0.0326 & -0.0117 & -0.0117 & -0.0117 & 0.3008 & -0.0117 \\
-0.0117 & -0.0117 & -0.0117 & -0.0326 & -0.0117 & -0.0117 & -0.0117 & -0.0326 & -0.0117 & -0.0117 & -0.0117 & -0.0326 & -0.0117 & -0.0117 & -0.0117 & 0.3008
\end{bmatrix}
$$

Table 6. The generalized inverse matrix L_3^+ of the information matrix of the square Ω_3

$$
L_3^+ = \begin{array}{cccccccccccccccc}
0.3320 & -0.0117 & -0.0117 & 0.0820 & -0.0430 & -0.0117 & -0.0117 & -0.0430 & -0.0430 & -0.0117 & -0.0117 & -0.0430 & -0.0430 & -0.0117 & -0.0117 & -0.0430 \\
-0.0117 & 0.3060 & -0.0169 & -0.0117 & -0.0117 & -0.0378 & -0.0065 & -0.0117 & -0.0117 & -0.0378 & -0.0169 & -0.0430 & -0.0117 & -0.0273 & -0.0065 & 0.0195 \\
-0.0117 & -0.0169 & 0.3060 & -0.0117 & -0.0117 & -0.0065 & -0.0378 & -0.0117 & -0.0117 & -0.0065 & -0.0273 & 0.0195 & -0.0117 & -0.0169 & -0.0378 & -0.0430 \\
0.0820 & -0.0117 & -0.0117 & 0.3320 & -0.0430 & -0.0117 & -0.0117 & -0.0430 & -0.0430 & -0.0117 & -0.0117 & -0.0430 & -0.0430 & -0.0117 & -0.0117 & -0.0430 \\
-0.0430 & -0.0117 & -0.0117 & -0.0430 & 0.3320 & -0.0117 & -0.0117 & 0.0820 & -0.0430 & -0.0117 & -0.0117 & -0.0430 & -0.0430 & -0.0117 & -0.0117 & -0.0430 \\
-0.0117 & -0.0378 & -0.0065 & -0.0117 & -0.0117 & 0.3060 & -0.0169 & -0.0117 & -0.0117 & -0.0273 & -0.0065 & 0.0195 & -0.0117 & -0.0378 & -0.0169 & -0.0430 \\
-0.0117 & -0.0065 & -0.0378 & -0.0117 & -0.0117 & -0.0169 & 0.3060 & -0.0117 & -0.0117 & -0.0169 & -0.0378 & -0.0430 & -0.0117 & -0.0065 & -0.0273 & 0.0195 \\
-0.0430 & -0.0117 & -0.0117 & -0.0430 & 0.0820 & -0.0117 & -0.0117 & 0.3320 & -0.0430 & -0.0117 & -0.0117 & -0.0430 & -0.0430 & -0.0117 & -0.0117 & -0.0430 \\
-0.0430 & -0.0117 & -0.0117 & -0.0430 & -0.0430 & -0.0117 & -0.0117 & -0.0430 & 0.3112 & -0.0117 & -0.0117 & 0.0195 & -0.0221 & -0.0117 & -0.0117 & 0.0195 \\
-0.0117 & -0.0378 & -0.0065 & -0.0117 & -0.0117 & -0.0273 & -0.0169 & -0.0117 & -0.0117 & 0.3060 & -0.0065 & 0.0195 & -0.0117 & -0.0378 & -0.0169 & -0.0430 \\
-0.0117 & -0.0169 & -0.0273 & -0.0117 & -0.0117 & -0.0065 & -0.0378 & -0.0117 & -0.0117 & -0.0065 & 0.3060 & 0.0195 & -0.0117 & -0.0169 & -0.0378 & -0.0430 \\
-0.0430 & -0.0430 & 0.0195 & -0.0430 & -0.0430 & 0.0195 & -0.0430 & -0.0430 & 0.0195 & 0.0195 & 0.0195 & 0.3320 & 0.0195 & -0.0430 & -0.0430 & -0.0430 \\
-0.0430 & -0.0117 & -0.0117 & -0.0430 & -0.0430 & -0.0117 & -0.0117 & -0.0430 & -0.0221 & -0.0117 & -0.0117 & 0.0195 & 0.3112 & -0.0117 & -0.0117 & 0.0195 \\
-0.0117 & -0.0273 & -0.0169 & -0.0117 & -0.0117 & -0.0378 & -0.0065 & -0.0117 & -0.0117 & -0.0378 & -0.0169 & -0.0430 & -0.0117 & 0.3060 & -0.0065 & 0.0195 \\
-0.0117 & -0.0065 & -0.0378 & -0.0117 & -0.0117 & -0.0169 & -0.0273 & -0.0117 & -0.0117 & -0.0169 & -0.0378 & -0.0430 & -0.0117 & -0.0065 & 0.3060 & 0.0195 \\
-0.0430 & 0.0195 & -0.0430 & -0.0430 & -0.0430 & -0.0430 & 0.0195 & -0.0430 & 0.0195 & -0.0430 & -0.0430 & -0.0430 & 0.0195 & 0.0195 & 0.0195 & 0.3320
\end{array}
$$

The frequency distribution of the variance of adjacency, $\mathrm{Var}(A_j)$ induced by each of the sixteen (16) treatments in each square are displayed in Table 7.

Table 7. The frequency distribution of the variance of adjacency, $\mathrm{Var}(A_j)$ induced by the various treatments in Ω_1, Ω_2, and Ω_3

Design	$\mathrm{Var}(A_j)$	**Frequency**
Ω_1	2.8125	12
	5.8125	4
Ω_2	1.3125	8
	5.8125	8
Ω_3	2.0625	8
	2.8125	2
	5.8125	6

From Table 7, it is obvious that; in Ω_1: twelve treatments induce a variance of adjacency equal to 2.8125, which is the minimum for this design, while a maximum variance of 5.8125 is being induced by its remaining four treatments; for Ω_2, there also exists two distinct variances of 1.3125 and 5.8125, each with multiplicity eight; and for Ω_3, there are three distinct variances, which are: 2.0625, 2.8125, and 5.8125, with multiplicities of eight, two, and six, respectively.

Also, from Table 7, we observe that, for each square, the maximum variance is the same, 5.8125. But for each of Ω_1 and Ω_2, there are two distinct variances: 2.8125 and 5.8125 for Ω_1; 1.3125 and 5.8125 for Ω_2. In Ω_1, the variance of 2.8125 is induced by the treatments A, α, a, B, β, b, C, γ, c, D, δ, and d, while the remaining treatments 1, 2, 3, and 4 induce the variance of 5.8125. Again, in Ω_2, the variance of 1.3125 is induced by the following treatments: α, a, β, b, γ, c, δ, and d, while A_1, A_2, B_1, B_2, C_1, C_2, D_1, and D_2 induce a variance of 5.8125.

Contrary to this, for Ω_3, there are three distinct variances: 2.0625, 2.8125, and 5.8125, which are induced by α, a, β, b, γ, c, δ, d; C, D; and A, 1, B, 2, 3, 4, respectively. Hence, Ω_1 and Ω_2 are to be preferred to Ω_3. Again, for Ω_2, the minimum value of this variance is 1.3125, which is less than that of Ω_1, 2.8125. Thus, Ω_2 is to be preferred to Ω_1.

6. Conclusion

We have ordered the semi-Latin squares, Ω_1, Ω_2, and Ω_3 considered in this work, in the order of preference, for purposes of experimentation. Based on our results in Section 5, it suffices to conclude that amongst the squares, Ω_2 is the most preferable one for experimentation, which is consistent with earlier results by Chigbu [9, 10] as well as Uto and Chigbu [23]; while Ω_1 is to be preferred to Ω_3, as found by Uto and Chigbu [23].

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