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THE DARBOUX VECTOR OF A NON-NULL CURVE IN $\mathbb{E}_1^4$

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Abstract

In this paper, we study the Darboux vector of a non-null curve in Minkowski 4-space $\mathbb{E}_1^4$. We also give a relation between the Darboux vector and harmonic curvatures. We find a relation between ccr-curves and harmonic curvatures of this curve. Further, we obtained some results for space-like and time-like vectors. We also show that if the ratios of the curvatures of a non-null curve are constant, then those of the ccr-curve are also constant.

1. Introduction

Let $X = (x_1, x_2, x_3, x_4)$ and $Y = (y_1, y_2, y_3, y_4)$ be two non-zero vectors in Minkowski 4-space $\mathbb{E}_1^4$. For $X, Y \in \mathbb{E}_1^4$

$$\langle X, Y \rangle = - x_1 y_1 + x_2 y_2 + x_3 y_3 + x_4 y_4,$$

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is called *Lorentzian inner product*. The couple $\{\mathbb{E}_1^4, \langle , \rangle\}$ is called *Lorentzian space*. Then the vector X of $\mathbb{E}_1^4$ is called (i) time-like if $\langle X, X\rangle < 0$, (ii) space-like if $\langle X, X\rangle > 0$ or $X = 0$, (iii) null (or light-like) vector if $\langle X, X\rangle = 0$, $X \neq 0$.

Similarly, an arbitrary curve $\alpha = \alpha(s)$ in $\mathbb{E}_1^4$ can be locally be space-like, time-like or null, if all of its velocity vectors $\alpha'(s)$ are, respectively, space-like, time-like or null. Also, recall the norm of a vector X is given by $\|X\| = \sqrt{|\langle X, X\rangle|}$. Therefore, X is a unit vector if $\langle X, X\rangle = \pm 1$. Next, vectors X, Y in $\mathbb{E}_1^4$ are said to be orthogonal if $\langle X, Y\rangle = 0$. The velocity of the curve α is given by $\|\alpha'\|$. Thus, a space-like or a time-like α is said to be parametrized by arclength function s, if $\langle \alpha', \alpha'\rangle = \pm 1$, [2].

2. Basic Definitions

Definition 1. Let $\alpha : I \to \mathbb{E}_1^4$ be a curve in $\mathbb{E}_1^4$ and k_1, k_2, k_3 be the Frenet curvatures of α. Then for the unit tangent vector $V_1 = \alpha'(s)$ over M the i-th e-curvature function m_i, $1 \leq i \leq 4$ is defined by

$$m_i = \left\{\begin{array}{ll} 0, & i = 1 \\ \dfrac{\varepsilon_1\varepsilon_2}{k_1}, & = 2 \\ \left[\dfrac{d}{dt}(m_{i-1}) + \varepsilon_{i-2}m_{i-2}k_{i-2}\right]\dfrac{\varepsilon_i}{k_{i-1}}, & 2 < i \leq 4 \end{array}\right\}, \tag{1}$$

where $\varepsilon_i = \langle V_i, V_i\rangle = \pm 1$.

Definition 2. Let $\alpha : I \to \mathbb{E}_1^4$ be a unit speed non-null curve. The curve α is called Frenet curve of osculating order d, $(d \leq 4)$ if its 4-th order derivatives $\alpha'(s), \alpha''(s), \alpha'''(s), \alpha^{iv}(s)$ are linearly independent and $\alpha'(s), \alpha''(s), \alpha'''(s), \alpha^{iv}(s), \alpha^{v}(s)$ are no longer linearly independent for all

$s \in I$. For each Frenet curve of order 4, one can associate an orthonormal 4-frame $\{V_1, V_2, V_3, V_4\}$ along α (such that $\alpha'(s) = V_1$) called the Frenet frame $k_1, k_2, k_3 : I \to \mathbb{R}$ called the Frenet curvatures, such that the Frenet formulas is defined in the usual way;

$$\begin{cases} \nabla_{V_1} V_1 = \varepsilon_2 k_1 V_2, \\ \nabla_{V_1} V_2 = -\varepsilon_1 k_1 V_1 + \varepsilon_3 k_2 V_3, \\ \nabla_{V_1} V_3 = -\varepsilon_2 k_2 V_2 + \varepsilon_4 k_3 V_4, \\ \nabla_{V_1} V_4 = -\varepsilon_3 k_3 V_3, \end{cases} \tag{2}$$

where $V_1, V_2, V_3,$ and V_4 are orthogonal vectors satisfying equations:

$$\langle V_1, V_1 \rangle = -1,$$

$$\langle V_i, V_i \rangle = 1, \quad (2 \leq i \leq 4),$$

and ∇ is the Levi-Civita connection of $\mathbb{E}_1^4$.

Definition 3. Let $X = (x_1, x_2, x_3, x_4)$, $Y = (y_1, y_2, y_3, y_4)$, and $Z = (z_1, z_2, z_3, z_4)$ be vectors in the space $\mathbb{E}_1^4$. The vector product in Minkowski space-time is defined with the determinant

$$X \wedge Y \wedge Z = -\begin{vmatrix} -e_1 & e_2 & e_3 & e_4 \\ x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \\ z_1 & z_2 & z_3 & z_4 \end{vmatrix},$$

where $e_1, e_2, e_3,$ and e_4 are coordinate direction vectors [5].

3. Harmonic Curvatures and Darboux Vector in $\mathbb{E}_1^4$

Definition 4. Let α be a non-null curve of osculating order 4. The harmonic functions

$$H_j : I \to \mathbb{R}, \quad 0 \leq j \leq 2,$$

defined by

$$H_0 = 0, \quad H_1 = \frac{k_1}{k_2},$$
$$H_j = \{\nabla_{v_1}(H_{j-1}) + \varepsilon_{j-2} H_{j-2} k_j\} \frac{\varepsilon_j}{k_{j+1}},$$

are called the harmonic curvatures of α. Here, k_1, k_2, k_3 are Frenet curvatures of α, ∇ is the Levi-Civita connection and $\varepsilon_j = \langle V_j, V_j \rangle = \pm 1$, [1].

Definition 5. Let α be a non-null curve of osculating order 4. Then α is called a general helix of *rank* 2 if

$$\sum_{i=1}^{2} H_i^2 = c,$$

holds, where $c \neq 0$ is a real constant.

We have the following result.

Corollary 1. *If* α *is a general helix of* $rank$ 2, *then*

$$H_1^2 + H_2^2 = c.$$

Proof. By the use of above definition we obtain.

Theorem 2. *Let* α *be a non-null curve of osculating order* 4 *in* $\mathbb{E}_1^4$, *then*

$$\begin{cases} \nabla_{V_1} V_1 = \dfrac{\varepsilon_1}{m_2} V_2, \\ \nabla_{V_1} V_2 = -\dfrac{\varepsilon_2}{m_2} V_1 + \dfrac{(m_2)'}{m_3} V_3, \\ \nabla_{V_1} V_3 = -\varepsilon_2 \varepsilon_3 \dfrac{(m_2)'}{m_3} V_2 + \left(\dfrac{m_3 (m_3)' + \varepsilon_2 \varepsilon_3 m_2 (m_2)'}{m_3 m_4} \right) V_4, \\ \nabla_{V_1} V_4 = \left(\dfrac{- \varepsilon_3 \varepsilon_4 m_3 (m_3)' - \varepsilon_2 \varepsilon_4 m_2 (m_2)'}{m_3 m_4} \right) V_3. \end{cases}$$

Here, the i-th *e-curvature function* m_i *and* $\varepsilon_i = \langle V_i, V_i \rangle = \pm 1, (1 \le i \le 4)$.

Proof. By using definition of the i-th e-curvature function m_i we get the result. □

Theorem 3. *Let* α *be a non-null curve of osculating order* 4 *in* $\mathbb{E}_1^4$, *then*

$$\begin{cases} \nabla_{V_1} V_1 = \varepsilon_2 k_2 H_1 V_2, \\ \nabla_{V_1} V_2 = -\varepsilon_1 k_2 H_1 V_1 + \varepsilon_3 \dfrac{k_1}{H_1} V_3, \\ \nabla_{V_1} V_3 = -\varepsilon_2 \dfrac{k_1}{H_1} V_2 + \varepsilon_2 \varepsilon_4 \dfrac{H_1'}{H_2} V_4, \\ \nabla_{V_1} V_4 = -\varepsilon_2 \varepsilon_3 \dfrac{H_1'}{H_2} V_3. \end{cases}$$

Here, H_1, H_2 *are harmonic curvatures of* α *and* $\varepsilon_i = \langle V_i, V_i \rangle = \pm 1$, $(1 \le i \le 4)$.

Proof. By using definition of harmonic curvatures, we get the result.

□

Theorem 4. *Let* $\alpha : I \to \mathbb{E}_1^4$ *be a non-null curve of osculating order* 4 *given the Frenet frame* $\{V_1, V_2, V_3, V_4\}$. *If* m_i, $2 \le i \le 4$, *are the* i-th *e-curvature functions and* H_i, $1 \le i \le 2$ *are the harmonic curvatures, then the following hold*:

$$\det(m_2', m_3', m_4') = 0 \Leftrightarrow \sum_{i=1}^{2} H_i^2 = constant.$$

Proof. The proof can seen by using the definitions of i-th e-curvature function m_i and harmonic curvatures H_i. □

Definition 6. Let α be a non-null curve of osculating order 4 in $\mathbb{E}_1^4$, with Frenet curvatures k_1, k_2, k_3. Let us denote

$$\begin{cases} a_0 = k_2 k_3, \\ a_1 = \varepsilon_1 \dfrac{k_1}{k_2} a_0, \quad (k_2 \neq 0), \\ a_2 = \varepsilon_2 \dfrac{k_2}{k_3} a_1, \quad (k_3 \neq 0). \end{cases}$$

The Darboux vector in $\mathbb{E}_1^4$ is defined by

$$D(s) = a_0 V_1 + a_1 V_2 + a_2 V_3,$$

where $\{V_1, V_2, V_3\}$ is the Frenet frame of α.

Lemma 5. *The derivative of the Darboux vector* $D(s)$ *is*

$$D'(s) = a_0' V_1 + a_1' V_2 + a_2' V_3,$$

[7].

Definition 7. The point $\alpha(s_0)$ is called Darboux vertex of α if the first derivative of the Darboux vector $D(s)$ is vanishing at that point [4].

Theorem 6. *Let* α *be a non-null curve of osculating order* 4 *in* $\mathbb{E}_1^4$, *with Frenet curvature* k_1 *and harmonic curvatures* H_1, H_2. *Let us denote*

$$\begin{cases} a_0 = \varepsilon_2 \dfrac{k_1 H_1'}{H_1 H_2}, \\ a_1 = \varepsilon_1 \varepsilon_2 \dfrac{k_1 H_1'}{H_2}, \\ a_2 = \varepsilon_1 \varepsilon_2 \dfrac{k_1^2}{H_1}, \end{cases}$$

where $H_1 \neq 0$ *and* $H_2 \neq 0$.

Proof. By using definition of harmonic curvatures, we get the result.

□

We obtain the following definition.

Definition 8. Let $\alpha : I \to \mathbb{E}_1^4$ be a non-null curve of osculating order 4. The harmonic functions

$$H_j : I \to \mathbb{R}, \quad 1 \leq j \leq 2,$$

defined by

$$\begin{cases} H_1 = \varepsilon_1 \dfrac{a_1}{a_0}, \\ H_2 = \varepsilon_1 \dfrac{a_2 H_1'}{a_0 k_1}, \end{cases}$$

where $\varepsilon_1 = \langle V_1, V_1 \rangle = \pm 1$; $a_0, a_1, a_2 \in R$ and k_1 is Frenet curvatures of α.

Theorem 7. *Let* α *be a non-null curve of osculating order* 4 *in* $\mathbb{E}_1^4$, *with Frenet curvatures* k_1, k_2, k_3. *The curve has a Darboux vertex at point* $\alpha(s)$ *if and only if*

$$\varepsilon_i \left(\frac{k_i}{k_{i+1}} \right)' = 0, \quad (1 \leq i \leq 2).$$

Corollary 8. *If* $\alpha : I \to \mathbb{E}_1^4$ *has a Darboux vertex at the point* $\alpha(s_0)$, *then* α *is a general helix of order* 3, [4].

4. Constant Curvature Ratios in $\mathbb{E}_1^4$

Definition 9. A curve $\alpha : I \to \mathbb{E}_1^4$ is said to have constant curvature ratios (that is to say, it is a ccr-curve) if all the quotients $\varepsilon_i\left(\frac{k_{i+1}}{k_i}\right)$ are constant $(k_i \neq 0)$. Here; k_i, k_{i+1}, $(1 \leq i \leq 2)$, are Frenet curvatures of α, and $\varepsilon_i = \langle V_i, V_i \rangle = \pm 1$, $(1 \leq i \leq 4)$.

Corollary 9. (a) *For* $i = 1$, *the ccr-curve is* $\varepsilon_2\varepsilon_3 \dfrac{m_2 (m_2)'}{m_3}$.

(b) *For* $i = 2$, *the ccr-curve is* $\varepsilon_2\varepsilon_3\varepsilon_4 \dfrac{m_3 (m_3)'}{(m_2)' m_4} + \varepsilon_4 \dfrac{m_2}{m_4}$.

Proof. The proof can be easily seen by using the definitions of i-th e-curvature function m_i and ccr-curve. □

Corollary 10. (a) *For* $i = 1$, *the ccr-curve is* $\dfrac{\varepsilon_1}{H_1}$.

(b) *For* $i = 2$, *the ccr-curve is* $\dfrac{H_1 H_1'}{H_2 k_1}$.

(c) *If the vector* V_1 *is time-like, then the ccr-curve is* $\dfrac{-1}{H_1}$, *where* $\varepsilon_1 = \langle V_1, V_1 \rangle = -1$.

(d) *If the vector* V_1 *is space-like, then the ccr-curve is* $\dfrac{1}{H_1}$, *where* $\varepsilon_1 = \langle V_1, V_1 \rangle = 1$.

Proof. The proof can be easily seen by using the definitions of harmonic curvature and ccr-curve. □

Corollary 11. *Let* $\alpha : I \to \mathbb{E}_1^4$ *is a ccr-curve. If*

$$\left(\varepsilon_1 \frac{1}{H_1}\right) = constant, \quad \left(\frac{H_1 H_1'}{H_2 k_1}\right) = constant,$$

then

$$\left(\varepsilon_1 \frac{1}{H_1}\right)' = 0, \quad \left(\frac{H_1 H_1'}{H_2 k_1}\right)' = 0.$$

Proof. The proof is obvious. □

Theorem 12. α *is a ccr-curve in* $\mathbb{E}_1^4 \Leftrightarrow \sum_{i=1}^{2} \varepsilon_i H_i^2 = constant.$

Proof. By using the definitions of a general helix of *rank* 2 and ccr-curve, this completes the proof of the theorem.

□

Theorem 13. *Let* $\alpha : I \to \mathbb{E}_1^4$ *is a non-null curve. Frenet frame* $\{V_1, V_2, V_3, V_4\}$ *and curvature functions* $k_1, k_2, k_3, (k_4 = 0)$. *If* $k_1 = 1$ *and* k_2, k_3 *are both constants, then*

$$\nabla_{V_1}^4 V_1 + \left(\varepsilon_1\varepsilon_2 + \varepsilon_2\varepsilon_3 \frac{k_1^2}{H_1^2}\right)\nabla_{V_1}^2 V_1 = 0.$$

Proof. $k_1 = 1$, from Equation (1), we have

$$\nabla_{V_1} V_1 = \varepsilon_2 V_2 \Rightarrow \nabla_{V_1}^2 V_1 = \varepsilon_2 \nabla_{V_1} V_2 \Rightarrow \nabla_{V_1}^3 V_1 = \varepsilon_2 \nabla_{V_1}^2 V_2 \Rightarrow \nabla_{V_1}^4 V_1 = \varepsilon_2 \nabla_{V_1}^3 V_2.$$

Since

$$\nabla_{V_1}^2 V_1 = -\varepsilon_1\varepsilon_2 V_1 + \varepsilon_2\varepsilon_3 k_2 V_3,$$

we have

$$\nabla_{V_1}^3 V_1 = -\varepsilon_1 V_2 - \varepsilon_3 k_2^2 V_2 + \varepsilon_2\varepsilon_3\varepsilon_4 k_2 k_3 V_4,$$

and

$$\nabla^4_{V_1} V_1 = \left(- \varepsilon_1\varepsilon_2 - \varepsilon_2\varepsilon_3 k_2^2\right)\nabla^2_{V_1} V_1 - \varepsilon_2\varepsilon_4 k_2 k_3^2 V_3, \tag{3}$$

where

$$\nabla^2_{V_1} V_2 = -\varepsilon_1\varepsilon_2 V_2 - \varepsilon_2\varepsilon_3 k_2^2 V_2 + \varepsilon_3\varepsilon_4 k_2 k_3 V_4,$$

$$\nabla^3_{V_1} V_2 = -\varepsilon_1 \nabla^2_{V_1} V_1 - \varepsilon_3 k_2^2 \nabla^2_{V_1} V_1 - \varepsilon_4 k_2 k_3^2 V_3,$$

and from Equation (3) $H_1 =$ constant, $H_1' = 0$ that is $k_3 = 0$. Thus we have

$$\nabla^4_{V_1} V_1 = (- \varepsilon_1\varepsilon_2 - \varepsilon_2\varepsilon_3 k_2^2)\nabla^2_{V_1} V_1,$$

or since $k_2 = \dfrac{k_1}{H_1}$, we obtain

$$\nabla^4_{V_1} V_1 + \left(\varepsilon_1\varepsilon_2 + \varepsilon_2\varepsilon_3 \frac{k_1^2}{H_1^2}\right)\nabla^2_{V_1} V_1 = 0.$$

□

Corollary 14. (i) *If the vector* V_1 *is time-like, then*

$$\nabla^4_{V_1} V_1 - \nabla^2_{V_1} V_1 + \frac{k_1^2}{H_1^2} \nabla^2_{V_1} V_1 = 0.$$

(ii) *If the vector* V_2 *is time-like, then*

$$\nabla^4_{V_1} V_1 - \nabla^2_{V_1} V_1 - \frac{k_1^2}{H_1^2} \nabla^2_{V_1} V_1 = 0.$$

(iii) *If the vector* V_3 *is time-like, then*

$$\nabla^4_{V_1} V_1 + \nabla^2_{V_1} V_1 - \frac{k_1^2}{H_1^2} \nabla^2_{V_1} V_1 = 0.$$

5. An Example

Example 1. Let us consider the following non-null curve in the space $\mathbb{E}_1^4$

$$\alpha(s) = \left(\sqrt{2}s,\ \sqrt{3},\ \sin s,\ \cos s\right).$$

$$V_1(s) = \alpha'(s) = \left(\sqrt{2},\ 0,\ \cos s,\ -\sin s\right),$$

where $\langle \alpha'(s), \alpha'(s) \rangle = -1$, which shows $\alpha(s)$ is an unit speed time-like curve. Thus $\|\alpha'(s)\| = 1.$ We express the following differentiations:

$$\begin{cases} \alpha''(s) = (0,\ 0,\ -\sin s,\ -\cos s), \\ \alpha'''(s) = (0,\ 0,\ -\cos s,\ \sin s), \\ \alpha^{iv}(s) = (0,\ 0,\ \sin s,\ \cos s). \end{cases}$$

Moreover, we have the first curvature, the second, the third curvature, harmonic curvature and i-th e-curvature function m_i of $\alpha(s)$ as

$$\begin{cases} k_1(s) = 1, \quad k_2(s) = \sqrt{2}, \quad k_3(s) = 0, \\ m_2 = -1, \quad m_3 = 0, \\ H_1 = \dfrac{1}{\sqrt{2}}, \quad H_2 = 0. \end{cases}$$

Now, we will calculate ccr-curve of $\alpha(s)$ in $\mathbb{E}_1^4$. If the vector V_1 is time-like, then $\varepsilon_1 = -1$

$$\begin{cases} \varepsilon_1 \dfrac{k_2}{k_1} = -\sqrt{2} = \text{constant}, \\ \varepsilon_2 \dfrac{k_3}{k_2} = 0 = \text{constant}. \end{cases}$$

Thus, $\alpha(s)$ is a ccr-curve in $\mathbb{E}_1^4$.

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