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THE ISOMORPHISM PROBLEM FOR CAYLEY GRAPHS ON THE GENERALIZED DICYCLIC GROUP

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Abstract

Let $Dic(a, b)$ be the generalized dicyclic group. In this paper, we show that if $Dic(a, b)$ has order $2ks$, $k > 1$, $k \in 2\mathbb{N} + 1$, $s \geq 1$ or 2^r, $r > 2$, then $Dic(a, b)$ is a Cayley isomorphic group with respect to graphs.

Keywords: Cayley graph, CI-graph, CI-group, generalized dicyclic group.

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1. Introduction

In 1967, Ádám [1] conjectured that any two Cayley graphs of $\mathbb{Z}_m$ are isomorphic if and only if they are isomorphic by an automorphism of $\mathbb{Z}_m$. A counter example to this conjecture was quickly found in 1970 by Elspas and Turner [6]. Posteriorly, Muzychuk completed the problem of determining which values of m have the property that any two Cayley graphs of $\mathbb{Z}_m$ are isomorphic if and only if they are isomorphic by an automorphism of $\mathbb{Z}_m$, proving that if m is square free, then $\mathbb{Z}_m$ [8] and $\mathbb{Z}_{2m}$ [9] have this property. The only other values of m with this property are 8 and 9 [2]. Ádáms conjecture was quickly generalized to the following problem.

Problem 1.1.

For which finite groups G is it true that any two Cayley graphs of G are isomorphic if and only if they are isomorphic by an automorphism of G?

A finite group G with this property will be called a Cayley isomorphic group (for brevity CI-group) with respect to graphs.

This problem are studied by many authors. Godsil [7] proved that Z_p^2 is a CI-group with respect to graphs for p a prime. Babai [3] showed that the nonabelian group of order $2p$ is a CI-group with respect to graphs, and the author [4] showed that the nonabelian group of order pq, p and q distinct primes, is a CI-group with respect to graphs if and only if $q = 2$ or 3. While some other results on the above problem are known, other than the above mentioned results of Muzychuk there are no known CI-groups with respect to graphs where the order of the group has more than three prime factors. In this paper, we show that if the generalized dicyclic group $Dic(a, b)$ has order $2ks, k > 1, k \in 2\mathbb{N} + 1, s \geq 1$ or 2^r, $r > 2$, then $Dic(a, b)$ is a Cayley isomorphic group with respect to graphs.

2. Preliminary Results

Definition 2.2. Let G be a finite group and let S be a subset of G which is closed under inversion and does not contain the identity. Then, the graph $\Gamma = \Gamma(G; S)$ given by $V(\Gamma) = G$ and $E(\Gamma) = \{(j, h) : j^{-1}h \in S\}$ is called Cayley graph.

Definition 2.3. Let G be a finite group. For $g \in G$, define $f_g : G \to G$ by $f_g(h) = gh$. Then $G_L = \{f_g : g \in G\}$ is itself a group, the left regular representation of G and is isomorphic to G.

Remark 2.4. Clearly if $\Gamma(G, S)$ is a Cayley graph on G, then $h^{-1}h' \in S$ and $(gh)^{-1}gh' \in S$ are equivalent statements, so that $f_g \in Aut(\Gamma)$, where $Aut(\Gamma)$ is the automorphism group of Γ. Therefore, $G_L \leq Aut(\Gamma)$.

We make some comments about of the normalizer $N_{Aut(\Gamma)}(G_L)$. It well known that for the normalizer of G_L in S_G, we have

$$N_{S_G}(G_L) = G_L \rtimes Aut(G).$$

Thus, we have

$$\begin{aligned} N_{Aut(\Gamma)}(G_L) &= (G_L \rtimes Aut(G)) \bigcap Aut(\Gamma) \\ &= G_L \rtimes (Aut(G) \bigcap Aut(\Gamma)) \qquad (2.1) \\ &= G_L \rtimes Aut(G, S), \end{aligned}$$

where $Aut(G, S) := \{\alpha \in Aut(G) \mid \alpha S = S\}$.

Let H be a subgroup of G_L. From (2.1), we deduce that

$$N_{Aut(\Gamma)}(H) = N_{G_L}(H) \rtimes Aut(G, H, S), \qquad (2.2)$$

where $Aut(G, H, S) := \{\alpha \in Aut(G, S) \mid \alpha H \alpha^{-1} = H\}$.

Definition 2.5. Let G be the finite group and let Γ be some Cayley graph of G. We shall say that Γ is a Cayley isomorphic graph (for brevity CI-graph) of G if given any Cayley graph Γ' of G such that Γ is isomorphic to Γ', then Γ and Γ' are isomorphic by some $\alpha \in Aut(G)$.

The following characterization of the CI-graph was proven by Babai and will be used in this paper.

Lemma 2.6 (Babai [3]). *For a Cayley graph Γ of a finite group G, the following are equivalent*:

(1) Γ *is a CI-graph of G*;

(2) *given a permutation $\phi \in S_G$ such that $\phi G_L \phi^{-1} \leq Aut(\Gamma)$, G_L and $\phi G_L \phi^{-1}$ are conjugate in $Aut(\Gamma)$.*

Other characterization of CI-graph we may find in the following lemma:

Lemma 2.7. *For a Cayley graph Γ of a finite group G, the following are equivalent*:

(1) Γ *is a CI-graph of G*;

(2) *given a permutation $\phi \in S_G$ such that $\phi h \in Aut(\Gamma)$ for some $h \in N_{S_G}(G_L)$.*

Proof. Assume that Γ is a CI-graph of G. According to Lemma 2.6, given a permutation $\phi \in S_G$ such that $\phi G_L \phi^{-1} \leq Aut(\Gamma)$, G_L and $\phi G_L \phi^{-1}$ are conjugate in $Aut(\Gamma)$, i.e., there exists $g \in Aut(\Gamma)$ such that

$$g G_L g^{-1} = \phi G_L \phi^{-1}. \tag{2.3}$$

From (2.3), we obtain

$$\phi^{-1} g G_L g^{-1} \phi = G_L. \tag{2.4}$$

Therefore, we may assert that $\phi^{-1} g = h \in N_{S_G}(G_L)$. The converse implication follows applying newly the last lemma. So we are done. □

Therefore, combining the Lemmas 2.6 and 2.7 we obtain:

Proposition 2.8. *For a Cayley graph* Γ *of a finite group* G, *the following are equivalent*:

(1) Γ *is a CI-graph of* G;

(2) *given a permutation* $\phi \in S_G$ *such that* $\phi G_L \phi^{-1} \leq Aut(\Gamma)$, G_L *and* $\phi G_L \phi^{-1}$ *are conjugate in* $Aut(\Gamma)$;

(3) *given a permutation* $\phi \in S_G$ *such that* $\phi h \in Aut(\Gamma)$ *for some* $h \in N_{S_G}(G_L)$.

Let R be a field of characteristic p or a complete discrete valuation ring with residue field of characteristic p. We recall that a minimal p-subgroup Q of G relative to which the indecomposable RG-module U is projective is called a vertex of U, and it is defined up to conjugacy in G. Moreover, an RQ-module Z for which U is a summand of $Ind_Q^G(Z)$ is called a source of U, and given the vertex Q it is defined up to conjugacy by elements of $N_G(Z)$. If $Z = R$ we say that U has trivial source.

The following lemma is well known.

Lemma 2.9. *A vertex of the trivial* RG-*module* R *is a Sylow* p-*subgroup of* G.

The following lemma will be used in this paper.

Lemma 2.10. *Let* G *be a finite group and let* P *be a* p-*subgroup of* G. *If* $P \in Syl_p(N_G(P))$, *then* $P \in Syl_p(G)$.

Proof. According to the last lemma, the p-subgroup P is a vertex of the trivial $RN_G(P)$-module. Since the Green correspondent of the trivial $RN_G(P)$-module is the trivial RG-module, applying again the Lemma 2.9 the result follows. □

3. Some Properties of the Group $Dic(a, b)$

Let $Dic(a, b) = \langle a, b : a^k = b^{ls}, bab^{-1} = a^u, a^{dk} = b^{dls} = 1\rangle$ be the finite group, where $k, s,$ and u are integers with $k > 1$ and $s \geq 1$. The positive integer d is a divisor of $u - 1$ and l is the multiplicative order of u modulo dk. The group is called generalized dicyclic group. Let $j = lsq + r', 0 \leq r' < ls$. Observe that for all element $g = b^j a^i (0 \leq i \leq dk - 1,$ $0 \leq j \leq dls - 1)$ we have

$$g = b^j a^i = b^j a^i b^{-j} b^j = a^{u^j i} b^j = a^{u^j i} b^{lsq + r'} = a^{u^j i + kq} b^{r'}.$$

Thus, for every element $g \in Dic(a, b)$, we may write $g = a^i b^j (0 \leq i \leq$ $dk - 1, 0 \leq j \leq ls - 1)$. Therefore, we may assert that $|Dic(a, b)| = dkls$.

Remark 3.11. Observe that when $u = -1$ and $s = 1$, the group is dihedral or general quaternion group according to $d = 1$ or $d = 2$.

Center of the Group

We denote the center of the group by $Z(Dic(a, b))$. Let d^* be the greatest common divisor of k and $\frac{u - 1}{d}$. In [5] was proved that

$$Z(Dic(a, b)) = \langle g \in Dic(a, b) : g = a^{\frac{k}{d^*}\beta} b^{l\delta} \rangle,$$

where $\beta = 0, \ldots, dd^* - 1, \delta = 0, \ldots, s - 1$. Hence, we have

$$|Z(Dic(a, b))| = dd^* s. \tag{3.1}$$

Remark 3.12. Let $I_{nn}(Dic(a, b))$ be inner automorphisms group. Then, from (3.1), we have $|I_{nn}(Dic(a, b))| = k'l, k' = \frac{k}{d^*}$.

Proposition 3.13. *Let* $Dic(a, b)$ *be the generalized dicyclic group. Then for the conjugacy classes* $a^{iDic(a,b)}(\gcd(i, dk) = 1)$ *and* $b^{jDic(a,b)}(1 \leq j < ls,$ $\gcd(j, dls) = 1)$, *we have*

$$|a^{iDic(a,b)}| = l \text{ and } |b^{jDic(a,b)}| = k'(k' = \frac{k}{d^*},\ d^* = \gcd(k, \frac{u-1}{d})).$$

Proof. Let a^i with $(\gcd(i, dk) = 1)$. Then for any element $a^x b^y \in Dic(a, b)$ we have

$$(a^x b^y)a^i(a^x b^y)^{-1} = a^{iu^y}.$$

Since $0 \leq y \leq l$ the result follows. Furthermore,

$$(a^x b^y)b^j(a^x b^y)^{-1} = a^{x(1-u)}b^j.$$

We may assert that $(1-u)x = dd^*u'x(u' = \dfrac{1-u}{dd^*})$ by assumption. So we are done. □

Lemma 3.14. *Let* $Dic(a, b)$ *be the generalized dicyclic group.*

(1) *If* $\gcd(d, k) = 1$, *then* $|Aut(Dic(a, b))| = dk\varphi(dk)\varphi(dls)$, *where* φ *is the Euler's phi function.*

(2) *If* $\gcd(d, k) = d$, *then* $|Aut(Dic(a, b))| = dk\varphi(dk)\varphi(ls)$.

Proof. (1) We claim that in such case $b^j \notin b^{Dic(a,b)}(j \in \{1, \ldots, dls-1\}$, $\gcd(j, dls) = 1$. Moreover, we may see that any $\alpha \in Aut(Dic(a, b))$ is given by $\alpha(a) = a^i$, $\gcd(i, dk) = 1$ and $\alpha(b) = a^x b^j$, $x = 0, \ldots, dk-1$; $\gcd(j, dls) = 1$. Assume that $A = \{a^i : \gcd(i, dk) = 1\}$ and $B = \{a^x b^j : x = 0, \ldots, dk-1;\ \gcd(j, dls) = 1\}$. Thus, we may write $|A| = \varphi(dk)$ and $|B| = dk\varphi(dls)$. Since $\alpha(a) = a^i \in A$ and $\beta(b) = a^x b^j \in B$ the result follows.

(2) Here $b^j \notin b^{Dic(a,b)}$ if and only if $(j \in \{1, \ldots, ls - 1\}), \gcd(j, dls) = 1$.

Thus, proceeding as in the case (1) the assertion follows. □

Lemma 3.15. *Let* $Dic(a, b)$ *be the generalized dicyclic group. Then* $Aut(Dic(a, b))$ *is a solvable group.*

Proof. We check two cases.

Case I: $\gcd(k, d) = 1$

According to the last lemma, we have $|Aut(Dic(a, b))| = dk\varphi(dk)\varphi(dls)$. Let H be a subgroup of $Aut(Dic(a, b))$ given by $H = \langle \alpha \in Aut(Dic(a, b)) |\alpha(a) = a, \alpha(b) = b^j (\gcd(j, dls) = 1, 1 \leq j \leq dls - 1\rangle$. We claim that H is abelian group of order $\varphi(dls)$. We consider the subgroup $H_1 = \langle \beta \in Aut(Dic(a, b))|\beta(a) = a^i (\gcd(i, dk) = 1, i \neq 1), \beta(b) = b^j (\gcd(j, dlk) = 1)$, $a^i \notin a^{Dic(a,b)}$ or $b^j \notin b^{Dic(a,b)}\rangle$. We may assert that H_1 is also abelian group whose order is given by $\frac{d\varphi(dk)}{l} q, q \in \{1, \ldots, l\}$. Hence $I_{nn}(Dic(a, b))H_1$ is normal subgroup of $Aut(Dic(a, b))$ of order $dk\varphi(dk)$. Thus, we may write

$$Aut(Dic(a, b)) = I_{nn}(Dic(a, b))H_1 \rtimes H. \tag{3.2}$$

Since $I_{nn}(Dic(a, b))$, H_1 and H are solvable groups, from (3.2) the result follows.

Case II: $\gcd(k, d) = d$

Proceeding as in the last case the assertion follows. □

4. Main Results

Theorem 4.16. *Let* $Dic(a, b)$ *be the generalized dicyclic group, where* $d = 1$, $l = 2$, $s \geq 1$ *and* $k > 1$ *is an odd number. Assume that* $\Gamma(Dic(a, b), S)$ *is a Cayley graph on* $Dic(a, b)$ *and that* 2^r *denote the* *highest power of 2 dividing* $|\langle f_b \rangle|$. *Then* $\langle f_a \rangle \times \langle f_{b^{2^r}} \rangle$ *is the largest normal* π*-subgroup of a Hall* π*-subgroup in* $Aut(\Gamma)$, *being* π *the set of all the odd prime divisors of* k *and s*.

Proof. Let us write G for $Aut(\Gamma)$ and we write H for $\langle f_a \rangle \times \langle f_{b^{2^r}} \rangle$. Firstly, we observe that $\langle f_{b^{2^r}} \rangle \leq Z(Dic(a,\ b)_L)$. Assume now that $P \leq H$ is a Sylow p-subgroup. Since P is a normal Sylow p-subgroup of $Dic(a,\ b)_L$, from (2.2), it follows that

$$N_G(P) = N_G(Dic(a,\ b)_L) = Dic(a,\ b)_L \rtimes Aut(Dic(a,\ b),\ S). \qquad (4.1)$$

By Lemma 3.15, it follows that $Aut(Dic(a,\ b))$ is a solvable group, so $Aut(Dic(a,\ b),\ S)$ is also solvable. Since $Dic(a,\ b)_L$ is solvable, from (4.1), we deduce that $N_G(Dic(a,\ b)_L)$ is solvable. Thus, the Hall π-subgroups of $N_G(Dic(a,\ b)_L)$ there exist. Let $\overline{H}$ be a fixed Hall π-subgroup of $Aut(Dic(a,\ b),\ S)$ and let $\overline{P}$ be a Sylow p-subgroup of $\overline{H}$. Then we may assert that the semidirect product $Q = P \rtimes \overline{P}$ is a Sylow p-subgroup of $N_G(P)$. Hence, we may write

$$N_G(Q) = P \rtimes N,$$

where N is the normalizer of $\overline{P}$ in $Aut(Dic(a,\ b),\ S)$.

Therefore $Q \in Syl_p(N_G(Q))$, so we may assert that Q is a Sylow p-subgroup of G by Lemma 2.10. Since the last statement hold for every Sylow p_i-subgroup ($p_i \in \pi,\ i = 1, \ldots, |\pi|$) of $H \rtimes \overline{H}$, we deduce that $H \rtimes \overline{H}$ is Hall π-subgroup of G, which is what we need to prove. □

We now give our main results in this paper.

Theorem 4.17. *Let* $Dic(a,\ b)$ *be the dicyclic generalized group, where* $d = 1$, $s \geq 1$, $u = k - 1$ *and* $k > 1$ *is an odd number. Then* $Dic(a,\ b)$ *is CI-group with respect to graphs*.

Proof. Assume that Γ is any Cayley graph on $Dic(a, b)$. Let us write G for $Aut(\Gamma)$. We will show that Γ is a CI-graph. Suppose that $\phi Dic(a, b)_L \phi^{-1} \leq G$, where $\phi \in S_{Dic(a, b)}$. According to Theorem 4.16, we may assert that $\langle f_a \rangle \times \langle f_{b^{2^r}} \rangle$ and $\phi \langle f_a \rangle \times \langle f_{b^{2^r}} \rangle \phi^{-1}$ are largest normal π-subgroup of a Hall π-subgroup of G respective, where π is the set of all the odd prime divisors of k and s. It well known that the Hall π-subgroups of a finite group (if they exist) are congugacy, so we may assert that for, some $g \in G$, we have $g\langle f_a \rangle \times \langle f_{b^{2^r}} \rangle g^{-1} = \phi \langle f_{a^i} \rangle \times \langle f_{(a^{2x}b^j)^{2^r}} \rangle \phi^{-1}$, $\gcd(k, i) = \gcd(s, j) = 1$, $x \in \{1, \ldots, k\}$ holds. Hence, we have

$$gag^{-1} = \phi a^i \phi^{-1} \text{ and } gb^{2^r} g^{-1} = \phi (a^{2x} b^j)^{2^r} \phi^{-1}. \tag{4.2}$$

From (4.2), it follows that $\phi^{-1} g = a^i \phi^{-1} g a^{-1}$. Therefore, we have

$$(\phi^{-1} g)^2 = (a^i (\phi^{-1} g) a^i)(a^{-1} (\phi^{-1} g) a^{-1}). \tag{4.3}$$

Thus, from (4.3), we obtain the following equality:

$$(\phi^{-1} g) a^{i+1} (\phi^{-1} g)^{-1} = a^{-(i+1)}. \tag{4.4}$$

Hence, from (4.4), we obtain

$$b(\phi^{-1} g) a^{i+1} (\phi^{-1} g)^{-1} b^{-1} = a^{i+1}. \tag{4.5}$$

Applying (4.2) secund part, we deduce that the equality (4.5) is true if and only if $b\phi^{-1} g \in \langle f_{b^2} \rangle$, so we may assert that $\phi \in G$. Therefore, according to Lemma 2.6, the Cayley graph Γ is CI-graph. So we are done. □

Theorem 4.18. *Let* $Dic(a, b)$ *be the dicyclic generalized group with* $|Dic(a, b)| = 2^r$, $r > 2$. *Then* $Dic(a, b)$ *is CI-group with respect to graphs.*

Proof. Let us write G for $Aut(\Gamma)$. Assume that Γ is a Cayley graph on $Dic(a, b)$. We will prove that Γ is CI-graph.

By assumption, we claim that $Aut(Dic(a, b))$ is a 2-group. Thus, we may assert that $Aut(Dic(a, b), S)$ is a 2-subgroup of $Aut(Dic(a, b))$. Hence, from (2.1), we deduce that the normalizer $N_G(Dic(a, b)_L)$ is a 2-group. We will prove that $N_G(Dic(a, b)_L)$ is a Sylow 2-subgroup of G.

Let R be a field of characteristic 2. Then we may assert that $RAut(Dic(a, b), S)$ is an indecomposable $RN_G(Dic(a, b)_L)$-module with vertex $Dic(a, b)_L$ and trivial source. Assume that P is a Sylow 2-subgroup of G such that $N_G(Dic(a, b)_L) \leq P$. Therefore, since RP is indecomposable applying the Green correspondence, we deduce the following holds:

$$P = N_G(Dic(a, b)_L).$$

Suppose that $\phi Dic(a, b)_L \phi^{-1} \leq G$, where $\phi \in S_{Dic(a,\, b)}$. We may assert that $N_G(\phi Dic(a, b)_L \phi^{-1}) \in Syl_p(G)$. Since all the Sylow 2-subgroups are conjugacy in G it follows that $Dic(a, b)_L$ and $\phi Dic(a, b)_L \phi^{-1}$ are conjugacy. Thus, applying the Lemma 2.6, we conclude that Γ is a CI-graph, which is what we need to prove. □

Remark 4.19. It is well known that the quaternion group is CI-group with respect to graphs. Observe that the quaternion group is a particular case of $Dic(a, b)$ under the conditions of the last theorem.

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