Journal of Mathematical Sciences: Advances and Applications
Volume 37, 2016, Pages 19-36
Available at http://scientificadvances.co.in
DOI: http://dx.doi.org/10.18642/jmsaa_7100121609

PRICING AMERICAN PUT OPTION WITH DIVIDENDS ON VARIATIONAL INEQUALITY

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Abstract

In this paper, considering the variational inequality model for American put option with dividends under Black-Scholes model, solving the variational equation by the finite difference method and the splitting method in time, numerical experiments have verified the effectiveness of the algorithm.

1. Introduction

Since 1973, the first time for exchange-traded futures in the United States, options market is developing very rapidly. Option theory was first put forward by Black and Scholes in [1], now it has become an important tool for financial risk management, as we know in [2], the core problem is the option pricing problem. There is no analytical solution for American options, it must use the numerical method and has challenges. Therefore, studying the numerical methods of the option pricing model has important theoretical significance and practical value in [3].

2010 Mathematics Subject Classification: F830.91, O211.63.
Keywords and phrases: Black-Scholes model, American put option, variational inequality.
Received December 25, 2015

Option pricing model numerical methods including lattice point method, partial differential equation method, and Monte Carlo stochastic simulation method. Lattice point method is simple, the algorithm is easy to implement, but the convergence speed is slow and the computation grows sharply as the number of nodes of time direction increases, stochastic simulation is not limited with dimension, but its weakness is that the slower convergence speed in [4]. At present, the research is based on partial differential equations of the numerical method, this method uses the finite difference method in [5], combining with the boundary conditions, the termination conditions and finally get the price of American options.

Based on results of domestic and foreign research in [6, 7], this paper consider the numerical solution of the variational inequality model for American put option with dividends, deep in studying on the splitting method in time, obtained the numerical algorithm with dividends, and the experimental results show that this method is a very effective in solving the American put option pricing model, and the splitting method is better than finite difference method.

2. The American Put Option Pricing Model

S, t, σ, r, q, T, and K represent price of the underlying asset at time, time, constant volatility, risk less interest rate in the bank, dividend rate, expiration time and strike price, the American put option $V(S, t)$ is suitable for the variational inequality model.

Seek $V(S, t) \in C_{\Sigma}^{1}$, it satisfies the following equality:

$$\begin{cases} \min\{-\mathcal{L}V,\ V-(K-S)^{+}\} = 0,\ (\Sigma) & (1) \\ V(S, T) = (K-S)^{+},\ 0 \le S < \infty, & (2) \\ V \to 0, \quad (S \to \infty), & (3) \end{cases}$$

for

$$\mathcal{L}V = \frac{\partial V}{\partial t} + \frac{\sigma^2}{2} S^2 \frac{\partial^2 V}{\partial S^2} + (r - q)S \frac{\partial V}{\partial S} - rV. \tag{4}$$

Here, we introduce the finite difference method and the splitting method in time to solve the American put option pricing with dividends.

3. The Finite Difference Method

In order to be able to solve the problem (1)-(3) on $\{\sum: 0 \le S < \infty, 0 \le t \le T\}$, we perform the following variable transformation:

$$x = \ln \frac{S}{K}, \tag{3.1}$$

$$v(x, t) = \frac{V(S, t)}{K}. \tag{3.2}$$

So the model of (1)-(3) become

$$\begin{cases} \min\{-\mathcal{L}_0 v, v - (1 - e^x)^+\} = 0, (x \in R, 0 \le t \le T), & (3.3) \\ v(x, T) = (1 - e^x)^+, (x \in R), & (3.4) \end{cases}$$

for

$$\mathcal{L}_0 v = \frac{\partial v}{\partial t} + \frac{\sigma^2}{2} \frac{\partial^2 v}{\partial x^2} + \left(r - \frac{\sigma^2}{2} - q\right) \frac{\partial v}{\partial x} - rv, \tag{3.5}$$

construct the grid on $(x \in R, 0 \le t \le T)$,

$$Q = \{(n\Delta t, j\Delta x) \mid 0 \le n \le N, j \in Z\},$$

where $\boldsymbol{Z}$ is the collection of natural number, $\Delta t = \frac{T}{N}$, $\Delta x > 0$.

Defined function at each grid node

$$v_j^n = v(j\Delta x, n\Delta t), \tag{3.6}$$

$$\varphi_j = \varphi(j\Delta x) = (1 - e^{j\Delta x})^+. \tag{3.7}$$

We take

$$\left(\frac{\partial v}{\partial t}\right)_{n+1,\,j} = \frac{v_{j+1}^n - v_j^n}{\Delta t},$$

$$\left(\frac{\partial v}{\partial x}\right)_{n+1,\,j} = \frac{v_{j+1}^{n+1} - v_{j-1}^{n+1}}{2\Delta x},$$

$$\left(\frac{\partial^2 v}{\partial x^2}\right)_{n+1,\,j} = \frac{v_{j+1}^{n+1} - 2v_j^{n+1} + v_{j-1}^{n+1}}{\Delta x^2},$$

Put them into (3.3)-(3.4), get the equation at $(j\Delta x,\ (n+1)\Delta t)$

$$\begin{cases} \min\left\{\begin{array}{l} -\dfrac{v_{j+1}^n - v_j^n}{\Delta t} - \dfrac{\sigma^2}{2}\dfrac{v_{j+1}^{n+1} - 2v_j^{n+1} + v_{j-1}^{n+1}}{\Delta x^2} - \\ \left(r - \dfrac{\sigma^2}{2} - q\right)\dfrac{v_{j+1}^{n+1} - v_{j-1}^{n+1}}{2\Delta x} + rv_j^n,\ v_j^n - \varphi_j \end{array}\right\} = 0,\ (0 \le n \le N-1,\ j \in \mathbf{z}), & (3.8) \\ v_j^N = \varphi_j. & (3.9) \end{cases}$$

Because

$$\min(A,\ B) = 0 \Leftrightarrow \min(\alpha A,\ B) = 0, \qquad (\alpha > 0),$$

and

$$\min(C - A,\ C - B) = 0 \Leftrightarrow C = \max(A,\ B).$$

In (3.8), take $\alpha = \Delta t$, get

$$\min\left\{(1 + r\Delta t)v_j^n - (1 - \omega)v_j^{n+1} - \alpha v_{j+1}^{n+1} - cv_{j-1}^{n+1},\ v_j^n - \varphi_j\right\} = 0, \tag{3.10}$$

here

$$\omega = \frac{\sigma^2 \Delta t}{\Delta x^2},$$

$$a = \frac{\omega}{2} + \frac{\Delta t}{2\Delta x}\left(r - \frac{\sigma^2}{2} - q\right),$$

$$c = \omega - a.$$

In (3.8), take $\alpha = \dfrac{1}{1 + r\Delta t}$, get

$$\min\left\{v_j^n - \frac{1}{1 + r\Delta t}\left[(1 - \omega)v_j^{n+1} + av_{j+1}^{n+1} + cv_{j-1}^{n+1}\right], v_j^n - \varphi_j\right\} = 0,$$

so

$$v_j^n = \max\left\{\frac{1}{1 + r\Delta t}\left[(1 - \omega)v_j^{n+1} + av_{j+1}^{n+1} + cv_{j-1}^{n+1}\right], \varphi_j\varphi_j\right\},$$

$$(0 \le n \le N - 1,\ j \in \mathbf{z}). \tag{3.11}$$

Theorem 3.1. *If*

$$\omega = \frac{\sigma^2\Delta t}{\Delta x^2} \le 1,$$

and

$$\frac{\mathbf{1}}{\boldsymbol{\sigma}^{\mathbf{2}}}\left|\boldsymbol{r} - \boldsymbol{q} - \frac{\boldsymbol{\sigma}^{\mathbf{2}}}{\mathbf{2}}\right|\Delta\boldsymbol{x} \le \mathbf{1}.$$

Then the difference scheme of (3.9), (3.11) *is convergent*, *namely*,

$$\lim_{\Delta t,\, \Delta x \to 0} v_\Delta(x, t) = v(x, t),$$

where $v_\Delta(x, t)$ *is the linear extension of* $v(j\Delta x, n\Delta t) = v_j^n$, $v(x, t)$ *is the viscosity solution of* (3.3), (3.4).

4. The Splitting Method

Consider the free boundary problem of the American put option model with dividends, $\{V(S, t), S(t)\}$ is the solution to (3.3)-(3.4) on $\sum_1$: $\{S(t) \le S < \infty, 0 \le t \le T\}$ with the following terminal and boundary conditions:

$$\begin{cases} \dfrac{\partial V}{\partial t} + \dfrac{\sigma^2}{2} S^2 \dfrac{\partial^2 V}{\partial S^2} + (r - q)S \dfrac{\partial V}{\partial S} - rV = 0, (\Sigma) & (4.1) \\ V(S(t), t) = K - S(t), & (4.2) \\ V_S(S(t), t) = -1, & (4.3) \\ V(S, T) = (K - S)^+, & (4.4) \\ V(S, t) \to 0, (S \to \infty), & (4.5) \\ S(T) = K. & (4.6) \end{cases}$$

Obviously, if we perform the following variable transformation:

$$\hat{S} = \frac{S}{K}, \quad \hat{V} = KV, \quad \hat{S}(t) = \frac{S(t)}{K},$$

Then we can unitizatize, namely, assuming that $K = 1$, we divide the interval $[0, T]$ into N parts with

$$0 = t_0 < t_1 < \cdots < t_N = T,$$

$$t_n = n\Delta t, \quad \Delta t = \frac{T}{N}.$$

At each node, define

$$V_n(S) = V(S, t_n) \text{与} S_n = S(t_n).$$

Make them suitable for a set of ordinary differential equation of free boundary problems by (4.1)-(4.6) on t discretization, namely, seek $\{V_n(S), S_n\}$, $(n = 0, 1, \cdots N)$, it satisfies the following equality:

$$\begin{cases} \dfrac{V_{n+1}(S) - V_n(S)}{\Delta t} + \dfrac{\sigma^2}{2} S^2 \dfrac{\partial^2 V_n}{\partial S^2} + (r - q)S \dfrac{\partial V_n}{\partial S} - rV_n = 0, \ (S_n \le S < \infty), & (4.7) \\ V_n(S_n) = 1 - S_n, & (4.8) \\ \dfrac{dV_n}{dS}(S_n) = -1, & (4.9) \\ V_n(S) \to 0, \ (S \to \infty), & (4.10) \\ V_N(S) = (1 - S)^+, & (4.11) \\ S_N = 1. & (4.12) \end{cases}$$

Here $0 \le n \le N - 1$.

The following is specific algorithm:

(1) Because $V_N(S) = (1 - S)^+$, $(0 \le S < \infty)$,

$$S_N = 1.$$

So, we get the value of $V_n(S)$ when $n = N$.

(2) From the induction process, we know $V_{n+1}(S)$, S_{n+1}, and $S_{n+1} < 1$, define

$$\hat{V}_{n+1}(S) = \begin{cases} V_{n+1}(S), & S_{n+1} \le S < \infty, \\ 1 - S, & 0 \le S < S_{n+1}. \end{cases}$$

Obviously, $\hat{V}_{n+1}(S) \in C^1_{[0,\infty)}$.

(3) To solve the following free boundary problem on $\{S_n \le S < \infty\}$,

$$\begin{cases} \mathcal{L}_\Delta V_n = \dfrac{\sigma^2}{2} S^2 \dfrac{d^2 V_n}{dS^2} + (r - q)S \dfrac{dV_n}{dS} - \left(r + \dfrac{1}{\Delta t}\right) V_n = \dfrac{-1}{\Delta t} \hat{V}_{n+1}(S), \\ \qquad\qquad (S_n \le S < \infty), & (4.13) \\ V_n(S_n) = 1 - S_n, & (4.14) \\ \dfrac{dV_n}{dS}(S_n) = -1, & (4.15) \\ V_n(S) \to 0, \ (S \to \infty). & (4.16) \end{cases}$$

We get $\{V_n, S_n\}$.

(4) To solve the differential equation of the free boundary problem (4.13)-(4.16), there are many ways to choose. Considering the particularity of this equation, can get a representation of the solution to Equation (4.13). We will introduce how to use method directly to solving V_n and S_n.

Firstly solve $V_{N-1}(S)$ and S_{N-1}. Equation (4.13) is a homogeneous second order ordinary differential equation. First, solve the general form of homogeneous equation, make

$$V_n(S) = S^{\alpha}.$$

Plug in (4.13), we get

$$K(\alpha) = \frac{\sigma^2}{2}\alpha(\alpha - 1) + (r - q)\alpha - \left(r + \frac{1}{\Delta t}\right) = 0. \tag{4.17}$$

Equation (4.17) is called the characteristic equation of ordinary differential equation (4.13), it has two roots:

$$\alpha_{\pm} = \gamma \pm \frac{\rho}{2}.$$

Here

$$\gamma = \frac{1}{2} - \frac{r - q}{\sigma^2},$$

$$\rho = \frac{2}{\sigma^2}\sqrt{\left(r - q - \frac{\sigma^2}{2}\right)^2 + 2\sigma^2\left(r + \frac{1}{\Delta t}\right)}.$$

So

$$\alpha_+ > 0 > \alpha_-.$$

Therefore, corresponding solution of (4.13) has the following form:

$$V(S) = d_1 S^{\alpha+} + d_2 S^{\alpha-},$$

from the nature of the best implementation boundary. Therefore, $S_{N-1} < 1$. The right part of (4.13) can be rewritten as the form of piecewise polynomial

$$\begin{cases} \mathcal{L}_\Delta V_{N-1} = 0, \quad (1 \le S < \infty), & (4.18) \\ \mathcal{L}_\Delta V_{N-1} = \dfrac{-1}{\Delta t}(1 - S) \quad (S_{N-1} \le S < 1), & (4.19) \\ V_{N-1}, V'_{N-1} \text{ continuous at } S = 1, & (4.20) \end{cases}$$

and suitable for boundary conditions (4.14)-(4.16).

Because the nonhomogeneous equation (4.19) have a special solution,

$$v = \frac{1}{1 + r\Delta t} - \frac{1}{\Delta t(1 + q\Delta t)} S.$$

So the solution of Equation (4.18)-(4.19) is

$$V_{N-1}(S) = \begin{cases} d_1^{(1)} S^{\alpha_+} + d_2^{(1)} S^{\alpha_-} + \dfrac{1}{1 + r\Delta t} - \dfrac{1}{\Delta t(1 + q\Delta t)} S, (S_{N-1} \le S \le 1), & (4.21) \\ d_1^{(2)} S^{\alpha_+} + d_2^{(2)} S^{\alpha_-}, (1 \le S \le \infty). & (4.22) \end{cases}$$

Here $d_i^{(m)}(i, m = 1, 2)$ and S_{N-1} are to be determined, they are made up of boundary conditions (4.14)-(4.16) and connection conditions (4.20). From (4.16), we know

$$d_1^{(2)} = 0. \tag{4.23}$$

From (4.20), we get

$$d_2^{(2)} = d_1^{(1)} S^{\alpha_+} + d_2^{(1)} S^{\alpha_-} + \frac{1}{1 + r\Delta t} - \frac{1}{\Delta t(1 + q\Delta t)}, \tag{4.24}$$

$$\alpha_- d_2^{(2)} = \alpha_+ d_1^{(1)} + \alpha_- d_2^{(1)} - \frac{1}{\Delta t(1 + q\Delta t)}. \tag{4.25}$$

From the free boundary conditions of (4.14) and (4.15), we know

$$d_1^{(1)} S_{N-1}^{\alpha_+} + d_2^{(1)} S_{N-1}^{\alpha_-} + \frac{1}{1 + r\Delta t} - \frac{1}{\Delta t(1 + q\Delta t)} = 0, \tag{4.26}$$

$$\alpha_+ d_1^{(1)} S_{N-1}^{\alpha_+ - 1} + \alpha_- d_2^{(1)} S_{N-1}^{\alpha_- - 1} - \frac{1}{\Delta t(1 + q\Delta t)} = -1. \tag{4.27}$$

From (4.24) and (4.25), we get

$$(\alpha_+ - \alpha_-) d_1^{(1)} = \frac{1}{\Delta t(1 + q\Delta t)} - \frac{\alpha_-}{\Delta t(1 + q\Delta t)} + \frac{\alpha_-}{1 + r\Delta t}.$$

Namely,

$$d_1^{(1)} = \frac{1}{\rho}\left(\frac{1}{\Delta t(1 + q\Delta t)} - \frac{\alpha_-}{\Delta t(1 + q\Delta t)} + \frac{\alpha_-}{1 + r\Delta t}\right). \tag{4.28}$$

For $(4.24) \times \alpha_- - (4.27) \times S_{N-1}$, we get the equations which is S_{N-1} suitable for

$$\rho d_1^{(1)} S_{N-1}^{\alpha_+} + \left(1 - \frac{1}{\Delta t(1 + q\Delta t)}\right) S_{N-1} - \frac{\alpha_-}{1 + r\Delta t} + \frac{\alpha_-}{\Delta t(1 + q\Delta t)} = 0. \tag{4.29}$$

So S_{N-1} can be work out.

Easy to know $0 < S_{N-1} < 1$, put $d_1^{(1)}$, S_{N-1} in (4.27), get

$$d_2^{(1)} = -\frac{\alpha_+}{\alpha_-} d_1^{(1)} S_{N-1}^{\alpha_+ - \alpha_-} + \frac{S_{N-1}^{1-\alpha_-}}{\Delta t(1 + q\Delta t)\alpha_-} - \frac{S_{N-1}^{1-\alpha_-}}{\alpha_-}. \tag{4.30}$$

Put $d_1^{(1)}$ and $d_2^{(1)}$ in (4.24), we get $d_2^{(2)}$. So we will work out $\{V_{N-1}(S), S_{N-1}\}$;

$$V_{N-1}(S) = \begin{cases} 1 - S, & (0 \le S \le S_{N-1}), \\ d_1^{(1)} S^{\alpha_+} + d_2^{(1)} S^{\alpha_-} + \dfrac{1}{1 + r\Delta t} - \dfrac{1}{\Delta t(1 + q\Delta t)} S, & (S_{N-1} \le S \le 1), \\ d_2^{(2)} S^{\alpha_-} & (1 \le S \le \infty). \end{cases}$$

Among them, the definition of $d_1^{(1)}$, $d_2^{(1)}$, $d_2^{(2)}$, and S_{N-1} in (4.28), (4.30), (4.23), and (4.29).

Lemma 4.1. *Considering nonhomogeneous ordinary differential equations*;

$$\mathcal{L}_\Delta(W) = \frac{\sigma^2}{2} S^2 \frac{d^2W}{dS^2} + (r-q)S\frac{dW}{dS} - \left(r + \frac{1}{\Delta t}\right)W = S^\omega \frac{1}{\Delta t}, \tag{4.31}$$

ω *is the single root of characteristic equation* (4.17), *namely*,

$$K(\omega) = \frac{\sigma^2}{2}\omega(\omega-1) + r\omega - \left(r + \frac{1}{\Delta t}\right) = 0. \tag{4.32}$$

The Equation (4.31) *have a special solution*

$$W(S) = \frac{1}{\Delta t K'(\omega)} S^\omega \ln S.$$

Now seek $\{V_{N-2}(S), S_{N-2}\}$. They are suitable for the following equations:

$$\begin{cases} \mathcal{L}_\Delta V_{N-2} = -\dfrac{1}{\Delta t} d_2^{(2)} S^{\alpha_-}, \quad (1 \le S \le \infty), & (4.33) \\ \mathcal{L}_\Delta V_{N-2} = -\dfrac{1}{\Delta t}\left(d_1^{(1)} S^{\alpha_+} + d_2^{(1)} S^{\alpha_-} + \dfrac{1}{1+r\Delta t} - \dfrac{1}{\Delta t(1+q\Delta t)} S\right), & \\ \qquad\qquad (S_{N-1} \le S \le 1), & (4.34) \\ \mathcal{L}_\Delta V_{N-2} = -\dfrac{1}{\Delta t}(1-S), \quad (S_{N-2} \le S \le S_{N-1}), & (4.35) \\ V_{N-2}, V'_{N-2} \text{ are continuous at } S = 1, & (4.36) \\ V_{N-2}, V'_{N-2} \text{ are continuous at } S = S_{N-1}, & (4.37) \\ V_{N-2}(S_{N-2}) = 1 - S_{N-2}, & (4.38) \\ V_{N-2}(S_{N-2}) = -1, & (4.39) \\ V_{N-2}(S) \to 0, (S \to \infty). & (4.40) \end{cases}$$

Equation (4.35) has a special root

$$W_1(S) = \frac{1}{1+r\Delta t} - \frac{1}{\Delta t(1+q\Delta t)} S. \tag{4.41}$$

The Equations (4.33) and (4.34) have special roots as follows:

$$W_2(S) = -\frac{1}{\Delta t}\left[d_1^{(1)}\frac{1}{K'(\alpha_+)}S^{\alpha_+}\ln S + d_2^{(1)}\frac{1}{K'(\alpha_-)}S^{\alpha_-}\ln S\right]$$

$$+\frac{1}{(1+r\Delta t)^2} - \frac{1}{\Delta t(1+q\Delta t)}S, \tag{4.42}$$

$$W_3(S) = -\frac{1}{\Delta t}d_2^{(2)}\frac{1}{K'(\alpha_-)}S^{\alpha_-}\ln S. \tag{4.43}$$

So, on $\{S_{N-2} \le S < \infty\}$, the general roots of (4.33)-(4.35) can be written as

$$V_{N-2}(S) = \begin{cases} W_1(S) + \hat{d}_1^{(1)}S^{\alpha_+} + \hat{d}_2^{(1)}S^{\alpha_-} \ (S_{N-2} \le S \le S_{N-1}), & (4.44) \\ W_2(S) + \hat{d}_1^{(2)}S^{\alpha_+} + \hat{d}_2^{(2)}S^{\alpha_-} \ (S_{N-1} \le S \le 1), & (4.45) \\ W_3(S) + \hat{d}_1^{(3)}S^{\alpha_+} + \hat{d}_2^{(3)}S^{\alpha_-} \ (1 \le S \le \infty). & (4.46) \end{cases}$$

Use conditions (4.36), (4.37), (4.38), (4.39), and (4.40) to calculate $d_i^{(m)}$ $(m = 1, 2, 3,\ i = 1, 2)$ and S_{N-2}.

From (4.40), we get

$$\hat{d}_1^{(3)} = 0. \tag{4.47}$$

From (4.36), we get

$$W_2(1) + \hat{d}_1^{(1)} + \hat{d}_2^{(1)} = W_3(1) + \hat{d}_2^{(3)}, \tag{4.48}$$

$$W_2'(1) + \alpha_+\hat{d}_1^{(2)} + \alpha_-\hat{d}_2^{(2)} = W_3'(1) + \alpha_-\hat{d}_2^{(3)}. \tag{4.49}$$

Because $W_2(1) = \dfrac{1}{(1+r\Delta t)^2} - \dfrac{1}{\Delta t(1+q\Delta t)}$, $W_3(1) = 0$, so

$$\hat{d}_1^{(2)} = \frac{1}{\rho}\left\{\alpha_-\left[\frac{1}{(1+r\Delta t)^2} - \frac{1}{\Delta t(1+q\Delta t)}\right] - [W_2'(1) - W_3'(1)]\right\}. \tag{4.50}$$

From (4.37), we get

$$W_1(S_{N-1}) + \hat{d}_1^{(1)} S_{N-1}^{\alpha_+} + \hat{d}_2^{(1)} S_{N-1}^{\alpha_-}$$

$$= W_2(S_{N-1}) + \hat{d}_1^{(2)} S_{N-1}^{\alpha_+} + \hat{d}_2^{(2)} S_{N-1}^{\alpha_-}, \tag{4.51}$$

$$W_1'(S_{N-1}) + \alpha_+ \hat{d}_1^{(1)} S_{N-1}^{\alpha_+ - 1} + \alpha_- \hat{d}_2^{(1)} S_{N-1}^{\alpha_- - 1}$$

$$= W_2'(S_{N-1}) + \alpha_+ \hat{d}_1^{(2)} S_{N-1}^{\alpha_+ - 1} + \alpha_- \hat{d}_2^{(2)} S_{N-1}^{\alpha_- - 1}.$$

So,

$$\hat{d}_1^{(1)} = \hat{d}_1^{(2)} + \frac{\alpha_-[W_1(S_{N-1}) - W_2(S_{N-1})] - S_{N-1}[W_1'(S_{N-1}) - W_2'(S_{N-1})]}{\rho S_{N-1}^{\alpha_+}}. \tag{4.52}$$

From (4.38) and (4.39), we get

$$W_1(S_{N-2}) + \hat{d}_1^{(1)} S_{N-2}^{\alpha_+} + \hat{d}_2^{(1)} S_{N-2}^{\alpha_-} = 1 - S_{N-2},$$

$$W_1'(S_{N-2}) + \alpha_+ \hat{d}_1^{(1)} S_{N-2}^{\alpha_+ - 1} + \alpha_- \hat{d}_2^{(1)} S_{N-2}^{\alpha_- - 1} = -1. \tag{4.53}$$

Because $W_1(S) = \frac{1}{1 + r\Delta t} - \frac{1}{\Delta t(1 + q\Delta t)} S$, $W_1'(S) = -\frac{1}{\Delta t(1 + q\Delta t)}$, thus

$$\hat{d}_1^{(1)} S_{N-2}^{\alpha_+} = \frac{1}{\rho}\left[\alpha_-\left(\frac{1}{1 + r\Delta t} - \frac{1}{\Delta t(1 + q\Delta t)}\right) - S_{N-2}\left(1 - \frac{1}{\Delta t(1 + q\Delta t)}\right)\right]. \tag{4.54}$$

Plug (4.50), (4.52) into (4.54), we got S_{N-2} suitable for transcendental equation, $S_{N-2} < S_{N-1}$.

From (4.53), we can get $\hat{d}_2^{(1)}$, now we know $\hat{d}_1^{(2)}$, $\hat{d}_1^{(1)}$, $\hat{d}_2^{(1)}$, we can infer $\hat{d}_2^{(2)}$ from (4.51), put it in (4.48), we can get $\hat{d}_2^{(3)}$, now we have calculated all of the undetermined coefficients $\hat{d}_i^{(m)}(m = 1, 2, 3, i = 1, 2)$ and S_{N-2}. The inference process are as follows:

$$\hat{d}_1^{(3)} \to \hat{d}_1^{(2)} \to \hat{d}_1^{(1)} \to S_{N-2} \to \hat{d}_2^{(1)} \to \hat{d}_2^{(2)} \to \hat{d}_2^{(3)}.$$

Using backward induction process, if $V_{n+1}(S) \in C^1_{[0,\infty)}$ and $S_{n+1}(n \geq 0)$ are known when $t = t_{n+1}$, $V_{n+1}(S)$ can be divided into the following internals:

$$[0, S_{n+1}], \cdots, [S_{N-1}, 1], [1, \infty).$$

When $0 \leq S \leq S_{n+1}$, $V_{n+1}(S) = 1 - S = M_1(S)$.

On $[S_{n+1}, \infty)$, $V_{n+1}(S)$ is piecewise function.

$$V_{n+1}(S) = M_j(S),\ j = 2, \cdots, N - n + 1.\ [S_{n+j-1}, S_{n+j}].$$

Among them

$$S_N = 1,$$

$$S_{N+1} = \infty,$$

$$M_j(S) = \sum_{i=1}^{n-2} c_i^{(j)} S^{\partial_+} (\ln S)^i + \sum_{i=1}^{n-2} d_i^{(j)} S^{\partial_-} (\ln S)^i + e^{(j)} S + f^{(j)}.$$

Here $c_i^{(j)}$, $d_i^{(j)}$, $e^{(j)}$, $f^{(j)}$ are constant ∂_+, ∂_- are roots of (4.17), so, in order to find $V_n(S)$ and S_n when $t = t_n$, we should rewrite (4.13)-(4.16) as follows:

$$\begin{cases} \mathcal{L}_\Delta V_n = -\dfrac{1}{\Delta t} M_1(S), \quad (S_n \leq S \leq S_{n+1}), & (4.55) \\ \mathcal{L}_\Delta V_n = -\dfrac{1}{\Delta t} M_2(S), \quad (S_{n+1} \leq S \leq S_{n+2}), & (4.56) \\ \quad \cdots\cdots \\ \mathcal{L}_\Delta V_n = -\dfrac{1}{\Delta t} M_{N-n}(S), \quad (S_{N-1} \leq S \leq 1), & (4.57) \\ \mathcal{L}_\Delta V_n = -\dfrac{1}{\Delta t} M_{N-n+1}(S), & (4.58) \end{cases}$$

$$V_n(S),\ V_n'(S) \text{ are continuous at } S = S_{n+1}, \tag{4.59}$$

……

$$V_n(S),\ V_n'(S) \text{ are continuous at } S = 1, \tag{4.60}$$

$$V_n(S_n) = 1 - S_n, \tag{4.61}$$

$$V_n'(S_n) = -1, \tag{4.62}$$

$$V_n(\infty) = 0. \tag{4.63}$$

$$V_n(S) = \begin{cases} W_1(S) + \hat{d}_1^{(1)} S^{\alpha_+} + \hat{d}_2^{(1)} S^{\alpha_-} & (S_n \le S \le S_{n-1}), \\ W_{N-n+1}(S) + \hat{d}_1^{(N-n+1)} S^{\alpha_+} + \hat{d}_2^{(N-n+1)} S^{\alpha_-} & (1 \le S < \infty). \end{cases}$$

Here $W_i(S)(i = 1, \cdots, N - n + 1)$ are special roots of (4.55)-(4.58) whose right side has the form of $-\frac{1}{\Delta t} M_i(S)$, we can find $\hat{d}_j^{(m)}(m = 1, \cdots, N - n + 1;\ j = 1, 2)$ and $S = S_n$ from (4.59), (4.60), (4.61), (4.62), and (4.63) when $S = 1, S_{N-1}, \cdots, S_{n+1}$, then solve them in turn

$$\hat{d}_1^{(n+1)} \to \hat{d}_1^{(n)} \to \cdots \to \hat{d}_1^{(1)} \to S_n \to \hat{d}_2^{(1)} \to \hat{d}_2^{(2)} \to \cdots \to \hat{d}_2^{(n+1)}.$$

5. Numerical Examples

Consider numerical simulation of American put option pricing problem in [8], in (1)-(3), set $\sigma = 0.2$, $K = 100$, $T = 1$, r and q take the following two kinds of situations: 1) $r = 0.05$, $q = 0.05$; 2) $r = 0.1$, $q = 0.01$. In order to verify the numerical results of different algorithm, numerical results are shown in Figure 1 and Figure 2. Here t represents time, S represents stock price; P represents the value of option. Contrast result of the finite difference method and the splitting method is given in Figure 1. Figure 2 shows the 3D images about the value of option. The numerical results shows that splitting method can match the option price more accurately.

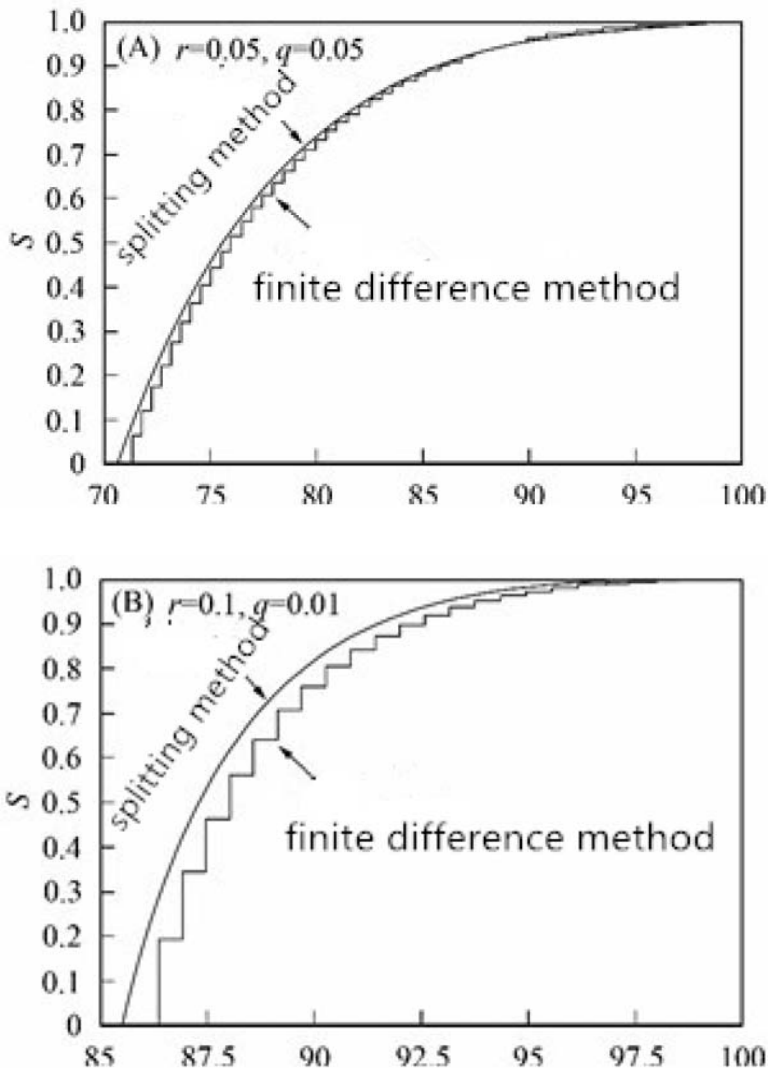

Figure 1. The best implementation border calculated by finite difference method and splitting method.

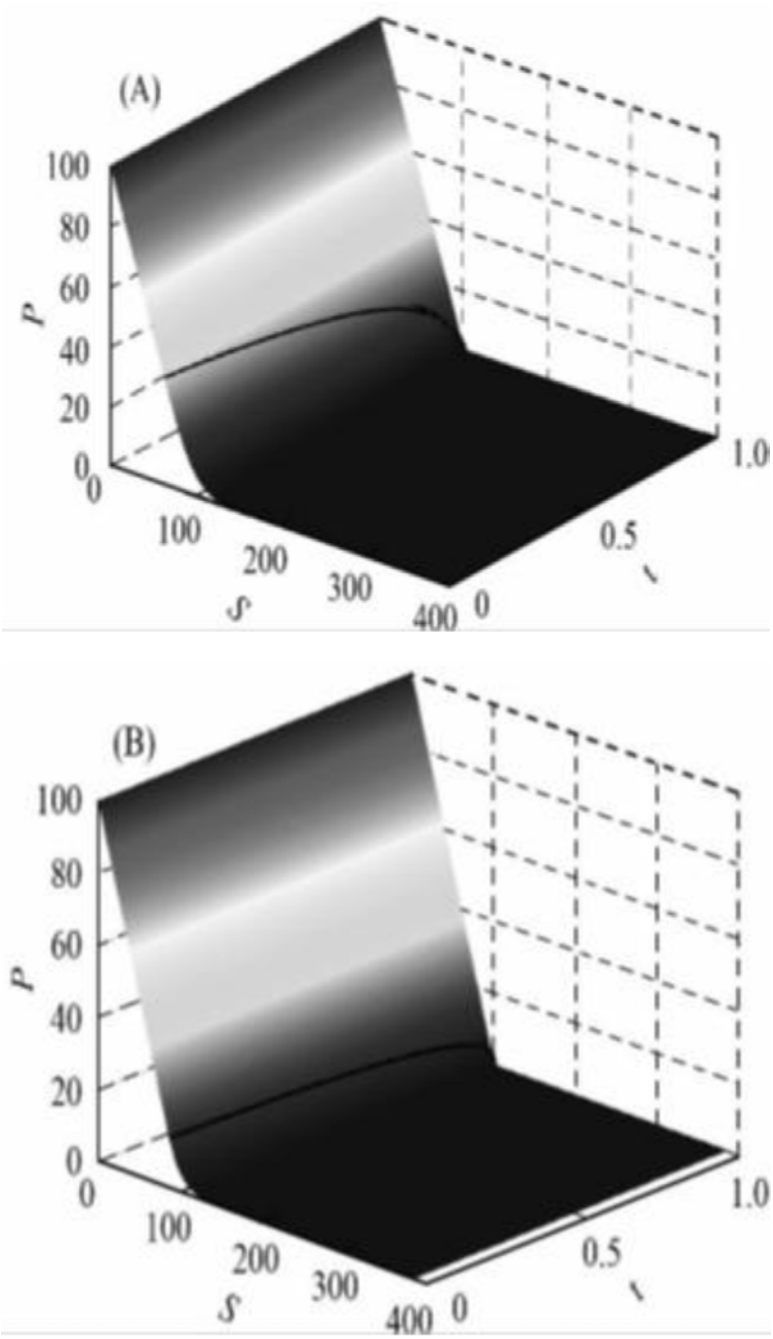

Figure 2. Three-dimension images about the value of option.

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