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SOME TYPES OF RICCI SOLITONS ON $(LCS)_n$-MANIFOLDS

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Abstract

The objective of the present paper is to study the some types of Ricci solitons on $(LCS)_n$-manifolds. We found the conditions of Ricci soliton on conformally flat, weakly symmetric, and weakly Ricci symmetric $(LCS)_n$-manifolds to be shrinking, steady, and expanding, respectively.

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1. Introduction

In 2003, Shaikh [35] introduced the notion of Lorentzian concircular structure manifolds (briefly, $(LCS)_n$-manifolds) with an example, which generalizes the notion of LP-Sasakian manifolds introduced by Matsumoto [25] and also by Mihai and Rosca [26]. Then Shaikh and Baishya ([38], [39]) investigated the applications of $(LCS)_n$-manifolds to the general theory of relativity and cosmology. The $(LCS)_n$-manifolds are also studied by Atceken et al. ([3], [4], [22]), Hui [21], Narain and Yadav [28], Prakasha [33], Shaikh and his co-authors ([36], [37], [40]-[44]) and many others.

In 1982, Hamilton [19] introduced the notion of Ricci flow to find a canonical metric on a smooth manifold. Then Ricci flow has become a powerful tool for the study of Riemannian manifolds, especially for those manifolds with positive curvature. Perelman ([31], [32]) used Ricci flow and its surgery to prove Poincare conjecture. The Ricci flow is an evolution equation for metrics on a Riemannian manifold defined as follows:

$$\frac{\partial}{\partial t} g_{ij}(t) = -2R_{ij}.$$

A Ricci soliton emerges as the limit of the solutions of the Ricci flow. A solution to the Ricci flow is called Ricci soliton if it moves only by a one parameter group of diffeomorphism and scaling. A Ricci soliton (g, V, λ) on a Riemannian manifold (M, g) is a generelization of Einstein metric such that [20]

$$\pounds_{V}g + 2S + 2\lambda g = 0, \tag{1.1}$$

where S is the Ricci tensor and $\pounds_V$ is the Lie derivative along the vector field V on M and λ is a real number. The Ricci soliton is said to be shrinking, steady and expanding according as λ is negative, zero, and positive, respectively.

During the last two decades, the geometry of Ricci solitons has been the focus of attention of many mathematicians. In particular, it has become more important after Perelman applied Ricci solitons to solve the long standing Poincare conjecture posed in 1904. In [47], Sharma studied the Ricci solitons in contact geometry. Thereafter Ricci solitons in contact metric manifolds have been studied by various authors, such as Bagewadi et al. ([1], [2], [5], [23]), Bejan and Crasmareanu [6], Blaga [7], Chandra et al. [12], Chen and Deshmukh [13], Deshmukh et al. [16], Nagaraja and Premlatta [27], Tripathi [51] and many others.

In 1926, Levy [24] proved that a second order parallel symmetric non-singular tensor in real space forms is proportional to the metric tensor. Then Sharma ([45], [46]) studied second order parallel tensor in Kaehler space of constant holomorphic sectional curvature as well as contact manifolds. Second order parallel tensor have been studied by various authors in different structure of manifolds. A tensor h of second order is said to be a parallel tensor if $\nabla h = 0$, where ∇ denotes the operator of covariant differentiation with respect to the metric tensor g. Recently, Chandra et al. [12] studied second order parallel tensors and Ricci solitons on $(LCS)_n$-manifolds.

Motivated by the above studies, the object of the present paper is to study Ricci soliton on $(LCS)_n$-manifolds. The paper is organized as follows. Section 2 is concerned with preliminaries. In Section 3 of this paper, we have studied conformally flat $(LCS)_n$-manifolds whose metric is Ricci soliton. We obtain the conditions of Ricci soliton of conformally flat $(LCS)_n$-manifold to be shrinking and expanding, respectively.

In [42], Shaikh and Binh studied weakly symmetric and weakly Ricci symmetric $(LCS)_n$-manifolds. Section 4 deals with the study weakly symmetric $(LCS)_n$-manifold whose metric is Ricci soliton. Section 5 is devoted with the study of weakly Ricci symmetric $(LCS)_n$-manifold, whose metric is Ricci soliton.

2. Preliminaries

An n-dimensional Lorentzian manifold M is a smooth connected paracompact Hausdorff manifold with a Lorentzian metric g, that is, M admits a smooth symmetric tensor field g of type $(0, 2)$ such that for each point $p \in M$, the tensor $g_p : T_pM \times T_pM \to \mathbb{R}$ is a non-degenerate inner product of signature $(-, +, \cdots, +)$, where T_pM denotes the tangent vector space of M at p and $\mathbb{R}$ is the real number space. A non-zero vector $v \in T_pM$ is said to be timelike (resp., non-spacelike, null, spacelike) if it satisfies $g_p(v, v) < 0$ (resp., $\leq 0, = 0, > 0$) [29].

Definition 2.1 ([52])**.** In a Lorentzian manifold (M, g), a vector field P defined by

$$g(X, P) = A(X),$$

for any $X \in \Gamma(TM)$, the section of all smooth tangent vector fields on M, is said to be a concircular vector field if

$$(\nabla_X A)(Y) = \alpha\{g(X, Y) + \omega(X)A(Y)\},$$

where α is a non-zero scalar and ω is a closed 1-form and ∇ denotes the operator of covariant differentiation with respect to the Lorentzian metric g.

Let M be an n-dimensional Lorentzian manifold admitting a unit timelike concircular vector field ξ, called the characteristic vector field of the manifold. Then we have

$$g(\xi, \xi) = -1. \tag{2.1}$$

Since ξ is a unit concircular vector field, it follows that there exists a non-zero 1-form η such that for

$$g(X, \xi) = \eta(X), \tag{2.2}$$

the equation of the following form holds:

$$(\nabla_X \eta)(Y) = \alpha\{g(X, Y) + \eta(X)\eta(Y)\}, \quad \alpha \neq 0, \tag{2.3}$$

that is,

$$\nabla_X \xi = \alpha[X + \eta(X)\xi],$$

for all vector fields $X, Y,$ where ∇ denotes the operator of covariant differentiation with respect to the Lorentzian metric g and α is a non-zero scalar function satisfies

$$\nabla_X \alpha = (X\alpha) = d\alpha(X) = \rho\eta(X), \tag{2.4}$$

ρ being a certain scalar function given by $\rho = -(\xi\alpha)$. If we put

$$\phi X = \frac{1}{\alpha}\nabla_X \xi, \tag{2.5}$$

then from (2.3) and (2.5), we have

$$\phi X = X + \eta(X)\xi, \tag{2.6}$$

from which it follows that ϕ is a symmetric (1, 1) tensor and called the structure tensor of the manifold. Thus, the Lorentzian manifold M together with the unit timelike concircular vector field ξ, its associated 1-form η and an (1, 1) tensor field ϕ is said to be a Lorentzian concircular structure manifold (briefly, $(LCS)_n$-manifold), [36]. Especially, if we take $\alpha = 1$, then we can obtain the LP-Sasakian structure of Matsumoto [25]. In a $(LCS)_n$-manifold $(n > 2)$, the following relations hold ([36], [38], [39], [40]):

$$\eta(\xi) = -1,\ \phi\xi = 0,\ \eta(\phi X) = 0,\ g(\phi X, \phi Y) = g(X, Y) + \eta(X)\eta(Y), \tag{2.7}$$

$$\phi^2 X = X + \eta(X)\xi, \tag{2.8}$$

$$S(X, \xi) = (n-1)(\alpha^2 - \rho)\eta(X), \tag{2.9}$$

$$R(X, Y)\xi = (\alpha^2 - \rho)[\eta(Y)X - \eta(X)Y], \tag{2.10}$$

$$R(\xi, Y)Z = (\alpha^2 - \rho)[g(Y, Z)\xi - \eta(Z)Y], \tag{2.11}$$

$$(\nabla_X \phi)Y = \alpha\{g(X, Y)\xi + 2\eta(X)\eta(Y)\xi + \eta(Y)X\}, \tag{2.12}$$

$$(X\rho) = d\rho(X) = \beta\eta(X), \tag{2.13}$$

$$R(X, Y)Z = \phi R(X, Y)Z + (\alpha^2 - \rho)\{g(Y, Z)\eta(X) - g(X, Z)\eta(Y)\}\xi, \tag{2.14}$$

$$S(\phi X, \phi Y) = S(X, Y) + (n-1)(\alpha^2 - \rho)\eta(X)\eta(Y), \tag{2.15}$$

for any vector fields X, Y, Z on M and $\beta = -(\xi\rho)$ is a scalar function, where R is the curvature tensor and S is the Ricci tensor of the manifold.

Definition 2.2. A $(LCS)_n$-manifold M is said to be η-Einstein if its Ricci tensor S of type $(0, 2)$ is of the form

$$S = pg + q\eta \otimes \eta,$$

where p and q are smooth functions on M.

Let (g, ξ, λ) be a Ricci soliton on a $(LCS)_n$-manifold M. From (2.3), we get

$$\frac{1}{2}(\pounds_\xi g)(X, Y) = \alpha\{g(X, Y) + \eta(X)\eta(Y)\}. \tag{2.16}$$

From (1.1) and (2.16), we have

$$S(X, Y) = -(\alpha + \lambda)g(X, Y) - \alpha\eta(X)\eta(Y), \tag{2.17}$$

which yields

$$QX = -(\alpha + \lambda)X - \alpha\eta(X)\xi, \tag{2.18}$$

$$S(X, \xi) = -\lambda\eta(X), \tag{2.19}$$

$$r = -\lambda n - (n-1)\alpha, \tag{2.20}$$

where Q is the Ricci operator, i.e., $g(QX, Y) = S(X, Y)$ for all X, Y and r is the scalar curvature of M.

We now recall the following:

Theorem 2.1 ([12])**.** *A second order parallel symmetric tensor on a* $(LCS)_n$*-manifold with* $\alpha^2 - \rho \neq 0$, *is a constant multiple of the metric tensor.*

Theorem 2.2 ([12]). *If the tensor field* $\pounds_V g + 2S$ *on a* $(LCS)_n$*-manifold with* $\alpha^2 - \rho \neq 0$ *is parallel for any vector field V, then* (g, V, λ) *is a Ricci soliton.*

3. Conformally Flat $(LCS)_n$-Manifolds whose Metric is Ricci Soliton

In differential geometry, the Weyl curvature tensor, named after Hermann Weyl, is a measure of the curvature of spacetime or, more generally, a pseudo-Riemannian manifold. Like the Riemann curvature tensor, the Weyl tensor expresses the tidal force that a body feels when moving along a geodesic. The Weyl tensor differs from the Riemann curvature tensor in that it does not convey information on how the volume of the body changes, but rather only how the shape of the body is distorted by the tidal force. The Ricci curvature, or trace component of the Riemann tensor contains precisely the information about how volumes change in the presence of tidal forces, so the Weyl tensor is the traceless component of the Riemann tensor. It is a tensor that has the same symmetries as the Riemann tensor with the extra condition that it be trace-free: metric contraction on any pair of indices yields zero.

In general relativity, the Weyl curvature is the only part of the curvature that exists in free space- a solution of the vacuum Einstein equation- and it governs the propagation of gravitational radiation through regions of space devoid of matter. More generally, the Weyl curvature is the only component of curvature for Ricci-flat manifolds and always governs the characteristics of the field equations of an Einstein manifold. In dimensions 2 and 3, the Weyl curvature tensor vanishes identically. In dimensions ≥ 4, the Weyl curvature is generally nonzero. If the Weyl tensor vanishes in dimension ≥ 4, then the metric is locally conformally flat: There exists a local coordinate system in which the metric tensor is proportional to a constant tensor. This fact was a key component of Nordström's theory of gravitation, which was a precursor of general relativity.

The Weyl tensor has the special property that it is invariant under conformal changes to the metric. For this reason, the Weyl tensor is also called the conformal tensor. It follows that a necessary condition for a Riemannian manifold to be conformally flat is that the Weyl tensor vanish. In dimensions ≥ 4, this condition is sufficient as well. In dimension 3, the vanishing of the Cotton tensor is a necessary and sufficient condition for the Riemannian manifold being conformally flat. Any 2-dimensional (smooth) Riemannian manifold is conformally flat, a consequence of the existence of isothermal coordinates. Conformal transformations of a Riemannian structures are an important object of study in differential geometry.

The conformal transformation on a $(LCS)_n$-manifold is a transformation under which the angle between two curves remains invariant. The Weyl conformal curvature tensor C of type (1, 3) of an n-dimensional Riemannian manifold $(LCS)_n$ $(n > 3)$ is defined by [15]

$$\begin{aligned} C(X, Y)Z = R(X, Y)Z - \frac{1}{n-2}[S(Y, Z)X - S(X, Z)Y \\ + g(Y, Z)QX - g(X, Z)QY] \\ + \frac{r}{(n-1)(n-2)}\{g(Y, Z)X - g(X, Z)Y\}. \end{aligned} \tag{3.1}$$

In [36], Shaikh studied some results on $(LCS)_n$-manifolds and proved that a conformally flat $(LCS)_n (n > 3)$ manifold is an η-Einstein manifold and its Ricci tensor is of the form

$$S(X, Y) = [\frac{r}{n-1} - (\alpha^2 - \rho)]g(X, Y) - [\frac{r}{n-1} - n(\alpha^2 - \rho)]\eta(X)\eta(Y). \tag{3.2}$$

From (3.2), we get

$$r = n(n-1)(\alpha^2 - \rho). \tag{3.3}$$

Suppose that h is a $(0, 2)$ symmetric parallel tensor field on a $(LCS)_n$-manifold with $\alpha^2 - \rho \neq 0$ such that

$$h(X, Y) = (\pounds_\xi g)(X, Y) + 2S(X, Y). \tag{3.4}$$

Then by virtue of Theorem 2.2, it follows that (g, ξ, λ) is a Ricci soliton. Using (3.2), (3.3), and (2.16) in (3.4), we get

$$h(X, Y) = 2[(n-1)(\alpha^2 - \rho) + \alpha]g(X, Y) + 2\alpha\eta(X)\eta(Y). \tag{3.5}$$

Putting $X = Y = \xi$ in (3.5), we get

$$h(\xi, \xi) = -2(n-1)(\alpha^2 - \rho). \tag{3.6}$$

Since (g, ξ, λ) is a Ricci soliton on a $(LCS)_n$-manifold M, then from (1.1), we have

$$h(X, Y) = -2\lambda g(X, Y), \tag{3.7}$$

and hence

$$h(\xi, \xi) = 2\lambda. \tag{3.8}$$

From (3.6) and (3.8), we get

$$\lambda = -(n-1)(\alpha^2 - \rho). \tag{3.9}$$

Since $n > 1$ and $(\alpha^2 - \rho) \neq 0$, we have $\lambda > 0$ or < 0 according as $(\alpha^2 - \rho) < 0$ or $(\alpha^2 - \rho) > 0$. Thus, we can state the following:

Theorem 3.1. *If the tensor field* $\pounds_\xi + 2S$, *on a conformally flat* $(LCS)_n$*-manifold* $(n > 3)$ *with* $\alpha^2 - \rho \neq 0$, *is parallel, then the Ricci soliton* (g, ξ, λ) *is shrinking and expanding according as* $\alpha^2 - \rho > 0$ *and* $\alpha^2 - \rho < 0$, *respectively*.

4. Weakly Symmetric $(LCS)_n$-Manifolds

The study of Riemann symmetric manifolds began with the work of Cartan [8]. A Riemannian manifold (M^n, g) is said to be locally symmetric due to Cartan [8] if its curvature tensor R satisfies the relation $\nabla R = 0$, where ∇ denotes the operator of covariant differentiation with respect to the metric tensor g. This condition of local symmetry is equivalent to the fact that at every point $P \in M$, the local geodesic symmetry $F(P)$ is an isometry [29]. The class of Riemann symmetric manifolds is very natural generalization of the class of manifolds of constant curvature.

During the last five decades, the notion of locally symmetric manifolds has been weakened by many authors in several ways to a different extent such as recurrent manifold by Walker [53], semisymmetric manifold by Szabó [48], pseudo-symmetric manifold in the sense of Deszcz [17], pseudo-symmetric manifold in the sense of Chaki [9], generalized pseudo-symmetric manifold by Chaki [11], weakly symmetric manifold by Selberg [34], and weakly symmetric manifold by Támassy and Binh [49]. It may be noted that the notion of weakly symmetric Riemannian manifolds by Selberg [34], is different and are not equivalent to that of Támassy and Binh [49].

As a proper generalization of pseudo-symmetric manifold by Chaki [9], Támassy and Binh [49] introduced the notion of weakly symmetric manifold and studied such structures on Sasakian manifolds and proved that such a structure does not always exist.

A non-flat Riemannian manifold $(M^n, g)(n > 2)$ is called a weakly symmetric manifold if the curvature tensor R of type $(0, 4)$ satisfies the condition

$$\begin{aligned}(\nabla_X R)(Y, Z, U, V) = {} & A(X)R(Y, Z, U, V) + B(Y)R(X, Z, U, V) \\ & + H(Z)R(Y, X, U, V) + D(U)R(Y, Z, X, V) \\ & + E(V)R(Y, Z, U, X), \qquad (4.1)\end{aligned}$$

for all vector fields $X, Y, Z, U, V \in \chi(M^n)$, where $A, B, H, D,$ and E are 1-forms (not simultaneously zero) and ∇ denotes the operator of covariant differentiation with respect to the Riemannian metric g. The

1-forms are called the associated 1-forms of the manifold and an n-dimensional manifold of this kind is denoted by $(WS)_n$. Moreover, it is to be noted that in a $(WS)_n$, $B = H$ and $D = E$ [14] and hence the defining condition (4.1) reduces to

$$\begin{aligned}(\nabla_X R)(Y, Z, U, V) = {} & A(X)R(Y, Z, U, V) + B(Y)R(X, Z, U, V) \\ & + B(Z)R(Y, X, U, V) + D(U)R(Y, Z, X, V) \\ & + D(V)R(Y, Z, U, X), \end{aligned} \tag{4.2}$$

where A, B, D are 1-forms (not simultaneously zero).

In [42], Shaikh and Binh studied weakly symmetric $(LCS)_n$-manifolds. A $(LCS)_n$-manifold is said to be weakly symmetric [42] if its curvature tensor R of type $(0, 4)$ satisfies the condition (4.2). It is proved that a weakly symmetric $(LCS)_n (n > 3)$ manifold is an η-Einstein manifold and its Ricci tensor is of the form

$$S(X, Y) = pg(X, Y) + q\eta(X)\eta(Y), \tag{4.3}$$

where $p = \dfrac{(\alpha^2 - \rho)\{(n-1)\alpha + B(\xi)\}}{\alpha + B(\xi)}$ and $q = -\dfrac{(n-1)(\alpha^2 - \rho)B(\xi)}{\alpha + B(\xi)}$, provided $\alpha + B(\xi) \neq 0$. In view of (2.16) and (4.3), it follows from (3.4) that

$$\begin{aligned} h(X, Y) = {} & 2\left[\frac{(\alpha^2 - \rho)\{(n-1)\alpha + B(\xi)\}}{\alpha + B(\xi)} + \alpha\right]g(X, Y) \\ & + 2\left[\frac{(n-1)(\alpha^2 - \rho)B(\xi)}{\alpha + B(\xi)} + \alpha\right]\eta(X)\eta(Y). \end{aligned} \tag{4.4}$$

Putting $X = Y = \xi$ in (4.4), we get

$$h(\xi, \xi) = \frac{2(\alpha^2 - \rho)[(n-2)B(\xi) - (n-1)\alpha]}{\alpha + B(\xi)}. \tag{4.5}$$

From (3.8) and (4.5), we get

$$\lambda = \frac{(\alpha^2 - \rho)[(n-2)B(\xi) - (n-1)\alpha]}{\alpha + B(\xi)}. \tag{4.6}$$

From (4.6), we get $\lambda > 0, = 0$ and < 0 according as

$$\frac{(\alpha^2 - \rho)[(n-2)B(\xi) - (n-1)\alpha]}{\alpha + B(\xi)} > 0, \quad \alpha = \frac{n-2}{n-1} B(\xi),$$

and

$$\frac{(\alpha^2 - \rho)[(n-2)B(\xi) - (n-1)\alpha]}{\alpha + B(\xi)} < 0.$$

This leads to the following:

Theorem 4.1. *If the tensor field* $\pounds_\xi + 2S$, *on a weakly symmetric* $(LCS)_n$*-manifold* $(n > 3)$ *with* $\alpha^2 - \rho \neq 0$ *is parallel, then the Ricci soliton* (g, ξ, λ) *is shrinking, steady and expanding according as*

$$\frac{(\alpha^2 - \rho)[(n-2)B(\xi) - (n-1)\alpha]}{\alpha + B(\xi)} > 0, \quad \alpha = \frac{n-2}{n-1} B(\xi),$$

and

$$\frac{(\alpha^2 - \rho)[(n-2)B(\xi) - (n-1)\alpha]}{\alpha + B(\xi)} < 0,$$

respectively.

5. Weakly Ricci Symmetric $(LCS)_n$-Manifolds

A Riemannian manifold is said to be Ricci symmetric if its Ricci tensor S of type $(0, 2)$ satisfies $\nabla S = 0$, where ∇ denotes the Riemannian connection. During the last five decades, the notion of Ricci symmetry has been weakened by many authors in several ways to a different extent such as Ricci recurrent manifold [30], Ricci semisymmetric manifold [48], pseudo Ricci symmetric manifold by Deszcz [18], and pseudo Ricci symmetric manifold by Chaki [10].

Extending all the above notions of Ricci symmetry, in 1993, Tamássy and Binh [50] introduced the notion of a weakly Ricci symmetric manifold. A Riemannian manifold $(M^n, g)(n > 2)$ is called weakly Ricci symmetric if its Ricci tensor S of type $(0, 2)$ is not identically zero and satisfies the condition

$$(\nabla_X S)(Y, Z) = A(X)S(Y, Z) + B(Y)S(Z, X) + D(Z)S(Y, X), \tag{5.1}$$

where A, B, and D are 1-forms (not simultaneously zero). Such an n-dimensional manifold is denoted by $(WRS)_n$. In [42], Shaikh and Binh studied weakly Ricci symmetric $(LCS)_n$-manifolds. A $(LCS)_n$-manifold is said to be weakly Ricci symmetric [42] if its Ricci tensor S of type $(0, 2)$ satisfies the condition (5.1). It is proved that a weakly Ricci symmetric $(LCS)_n (n > 3)$ manifold is an η-Einstein manifold and its Ricci tensor is of the form

$$S(X, Y) = \frac{(n-1)\alpha(\alpha^2 - \rho)}{\alpha + D(\xi)} g(X, Y) - \frac{(n-1)(\alpha^2 - \rho)D(\xi)}{\alpha + D(\xi)} \eta(X)\eta(Y), \tag{5.2}$$

provided $\alpha + D(\xi) \neq 0$.

In view of (2.16) and (5.2), it follows from (3.4) that

$$h(X, Y) = 2\left[\frac{(n-1)\alpha(\alpha^2 - \rho)}{\alpha + D(\xi)} + \alpha\right] g(X, Y) + 2\left[\alpha - \frac{(n-1)(\alpha^2 - \rho)D(\xi)}{\alpha + D(\xi)}\right]\eta(X)\eta(Y). \tag{5.3}$$

Putting $X = Y = \xi$ in (5.3), we get

$$h(\xi, \xi) = -2(n-1)(\alpha^2 - \rho). \tag{5.4}$$

From (3.8) and (5.4), we get

$$\lambda = -(n-1)(\alpha^2 - \rho). \tag{5.5}$$

Since $n > 1$ and $\alpha^2 - \rho \neq 0$, we have $\lambda > 0$ or < 0 according as $\alpha^2 - \rho > 0$ and $\alpha^2 - \rho < 0$. This leads to the following:

Theorem 5.1. *If the tensor field* $\pounds_\xi + 2S$, *on a weakly symmetric* $(LCS)_n$*-manifold* $(n > 3)$ *with* $\alpha^2 - \rho \neq 0$ *is parallel, then the Ricci soliton* (g, ξ, λ) *is shrinking and expanding according as* $\alpha^2 - \rho > 0$ *and* $\alpha^2 - \rho < 0$, *respectively*.

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