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DEDEKIND'S ARITHMETIC FUNCTION AND PRIMITIVE FOUR SQUARES COUNTING FUNCTIONS

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Abstract

We define summatory odd and even Dedekind arithmetic functions and obtain their first order asymptotic estimates. Based on this, we determine an asymptotic formula for the cumulative number of primitive four squares representations, which compares with Landau's original formula for arbitrary four squares representations. The introduced arithmetic functions are also used to study the asymptotic behaviour of the cumulative number of distinct primitive Pythagorean quintuples. A conjecture is formulated at the end of the paper.

1. Introduction

The cornerstone and most important result in additive number theory is the theorem of Lagrange [7], which states that that every natural number can be written as a sum of four squares (e.g., Nathanson

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[10], Subsection 1.2). This result was already conjectured in 1621 by Bachet (1581-1638). More generally, one is interested in the representations of a natural number n as a sum of $k \geq 2$ squares such that $x_1^2 + x_2^2 + \cdots + x_k^2 = n$. The number of such representations, counting zeros, permutations and sign changes, is denoted by $r_k(n)$. Similarly, the abbreviation $R_k(n)$ denotes the number of *primitive* representations of n as a sum of $k \geq 2$ squares counting zeros, permutations and sign changes. Given a formula for $r_k(n)$, one obtains from it a formula for $R_k(n)$ through Möbius inversion of the basic identity (e.g., Grosswald [3], Theorem 1, Subsection 1.1)

$$r_k(n) = \sum_{d^2 | n} R_k\left(\frac{n}{d^2}\right). \tag{1.1}$$

With this recipe, Cooper and Hirschhorn [2] obtain a wide variety of formulas for $R_k(n)$ within the range $2 \leq k \leq 8$ for any n, and the range $9 \leq k \leq 12$ for certain values of n.

One often considers the summatory functions of these counting functions defined by

$$n_k(x) = \sum_{n \leq x} r_k(n), \quad N_k(x) = \sum_{n \leq x} R_k(n). \tag{1.2}$$

Their interest is due to the fact that the asymptotic behaviour of (1.2) divided by x yields the average order of $r_k(n)$, respectively, $R_k(n)$ in the range $1 \leq n \leq x$. For example, as already known to Gauss, one has the formula (e.g., Krätzel [6], Satz 5.37; Narkiewicz [9], Corollary 1, p. 196)

$$n_2(x) = \pi \cdot x + O(\sqrt{x}). \tag{1.3}$$

The error term of this approximation has been improved to $O(x^{1/3})$ by Sierpinski [14]. Elementary derivations of the weaker results $O(x^{1/3}(\log x)^{2/3})$ and $O(x^{1/3} \log x)$ are found in Krätzel [6], Satz 6.26, respectively, Narkiewicz [9], Theorem 5.12, p. 203 (for further information consult Grosswald [3], Subsection 2.2). Walfisz [16] studies this function for $k \geq 4$, for which one has

$$n_k(x) = \frac{(\pi \cdot x)^{k/2}}{\Gamma(1 + k / 2)} + E_k(x), \tag{1.4}$$

with $E_k(x) = o(x^{k/2})$. The best estimate of the error has also been studied. For $k \geq 5$, one has $E_k(x) = O(x^{k/2-1})$ but the estimate $E_k(x) = o(x^{k/2-1})$ is not true (see Narkiewicz [9], p. 205; Krätzel [6], Satz 6.34). For $k = 4$, the situation is different and one has (e.g., Krätzel [6], Satz 6.33)

$$n_4(x) = \frac{1}{2}\pi^2 \cdot x^2 + O(x \log x), \tag{1.5}$$

whose error term can be improved to $O(x \log x (\log \log x)^{-1})$ but not to $o(x \log \log x)$ (see Narkiewicz [9], p. 205; Krätzel [6], Satz 6.35). Originally, a first version of (1.5) was obtained by Landau [8] (see Grosswald [3], Subsection 3.5). On the other hand, information about the asymptotic behaviour of $N_k(x)$ seems not readily available. As a main application of the properties of Dedekind's arithmetic function we derive in Theorem 3.1 the estimate

$$N_4(x) = \frac{45}{\pi^2} \cdot x^2 + O(x \log x), \tag{1.6}$$

which should be compared with (1.5). The ratio of the leadings terms (1.6) to (1.5) equals 0.92394. Therefore, there are only marginally less primitive representations than arbitrary ones.

To show (1.6), one uses the fact that $R_4(n)$ is a certain multiple of the Dedekind arithmetic or psi function defined and denoted by (as always the letter p denotes a prime number)

$$\psi(n) = n \cdot \prod_{p|n} (1 + \frac{1}{p}). \tag{1.7}$$

Besides estimates of the summatory psi function $\Psi(x) = \sum_{n \le x} \psi(n)$, one needs estimates for the odd and even summatory psi functions defined by $\Psi^o(x) = \sum_{\text{odd } n \le x} \psi(n)$ and $\Psi^e(x) = \sum_{\text{even } n \le x} \psi(n)$, respectively. The required preliminaries about them are provided in Section 2.

A further application of the Dedekind odd and even summatory psi functions concerns the cumulative number of primitive solutions of the Diophantine equation

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 = n^2, \tag{1.8}$$

below a limit x taking into account zeros, permutations and sign changes. This counting function is denoted by $NS_4(x) = \sum_{n \le x} R_4(n^2)$ and its first order asymptotic behaviour is derived in Theorem 4.1. It is related to the number of distinct primitive Pythagorean quintuples without zeros, for which an exact formula is found in Hürlimann [5], Theorem 5.1. We conclude with a perspective on possible future developments.

2. Summatory Odd and Even Psi Functions and their First Order Asymptotic Behaviour

The Dedekind integer sequence $\psi(n) = n \cdot \prod_{p|n} (1 + \frac{1}{p})$ is sequence A001615 in Sloane [15], which is currently tabulated for all $1 \le n \le 10^4$. It is closely related to Euler's totient function $\varphi(n) = n \cdot \prod_{p|n} (1 - \frac{1}{p})$, and it satisfies the Dirichlet convolution product $\psi = Id * \mu$, with μ the Möbius function, i.e., the identity (e.g., Bordellès [1], Equation (3), p. 3)

$$\psi(n) = n \cdot \sum_{d|n} \frac{|\mu(d)|}{d}. \tag{2.1}$$

We begin with a basic result on the summatory psi function, which is sequence A173290 in Sloane's OEIS.

Lemma 2.1. *The summatory Dedekind arithmetic function satisfies the asymptotic relation*

$$\Psi(x) = \frac{15}{2\pi^2} \cdot x^2 + O(x \log x). \tag{2.2}$$

Proof. We follow Pérez Herrero [12], which mimics the proof of Theorem 330 on the summatory Euler function $\Phi(x) = \sum_{n \le x} \varphi(n)$ in Hardy and Wright [4]. Using (2.1), one has

$$\Psi(x) = \sum_{n \le x} n \cdot \sum_{d|n} \frac{|\mu(d)|}{d} = \sum_{dd' \le x} d' \cdot |\mu(d)| = \sum_{d \le x} |\mu(d)| \cdot \sum_{d' \le \lfloor x/d \rfloor} d' = \frac{1}{2} \cdot \sum_{d \le x} |\mu(d)|$$

$$\cdot \{\lfloor x / d \rfloor^2 + \lfloor x / d \rfloor\}$$

$$= \frac{1}{2} \cdot \sum_{d \le x} |\mu(d)| \cdot \{x^2 / d^2 + O(x / d)\} = \frac{1}{2} x^2 \cdot \sum_{d \le x} \frac{|\mu(d)|}{d^2} + O(x \cdot \sum_{d \le x} \frac{1}{d})$$

$$= \frac{1}{2} x^2 \cdot \sum_{d \ge 1} \frac{|\mu(d)|}{d^2} + O(x^2 \cdot \sum_{d \ge x+1} \frac{1}{d^2}) + O(x \log x) = \frac{15}{2\pi^2} \cdot x^2 + O(x)$$

$$+ O(x \log x),$$

which implies (2.2). Note that the last step makes use of the fact $\sum_{d \ge 1} \frac{|\mu(d)|}{d^2} = \frac{15}{\pi^2}$ (Sloane's OEIS A082020, Ramanujan [13], Weisstein [17], Equation (12)). □

Remark 2.1. An improved error term for (2.2) is found in Bordellès [1], p. 15:

$$\Psi(x) = \frac{15}{2\pi^2} \cdot x^2 + O(x(\log x)^{2/3} (\log \log x)^{4/3}). \tag{2.3}$$

For use in the next section, we need estimates for the odd and even summatory psi functions.

Proposition 2.1. *The summatory odd and even Dedekind arithmetic functions satisfy the asymptotic relations*

$$\Psi^o(x) = \frac{3}{\pi^2} \cdot x^2 + O(x \log x), \quad \Psi^e(x) = \frac{9}{2\pi^2} \cdot x^2 + O(x \log x). \tag{2.4}$$

Proof. In a first step, let us express the even $\Psi^e(x)$ as function of the odd $\Psi^o(x)$. If $2 \le n \le x$ is even, then there exists $1 \le k \le \lfloor \ln x / \ln 2 \rfloor$ such that $n = 2^k \cdot s$ with s odd and $1 \le s \le 2^{-k} \cdot x$. It follows that

$$\Psi^e(x) = \sum_{k=1}^{\lfloor \ln x / \ln 2 \rfloor} \left\{ \sum_{\text{odd } s \le 2^{-k} x} \psi(2^k s) \right\} = \frac{3}{2} \cdot \sum_{k=1}^{\lfloor \ln x / \ln 2 \rfloor} 2^k \cdot \psi^o(2^{-k} \cdot x).$$

Now, suppose that $\Psi^o(x) = cx^2 + O(x \log x)$ for some constant c. Using that $\Psi(x) = \Psi^o(x) + \Psi^e(x)$, Lemma 2.1, and the preceding identity, one must have

$$\frac{15}{2\pi^2} \cdot x^2 \sim cx^2 + \frac{3}{2} \left(\sum_{k=1}^{\lfloor \ln x / \ln 2 \rfloor} 2^{-k} \right) cx^2 \quad (x \to \infty),$$

which holds exactly when $\frac{5}{2} c = \frac{15}{2\pi^2}$, or $c = \frac{3}{\pi^2}$. The result follows. □

The exact and asymptotic estimates of these summatory psi functions do not differ very much. Table 2.1 illustrates for $x \le 10^4$ (calculation based on A001615). The (irregular) decrease of the relative percentage error shows the increasing goodness-of-fit of the asymptotic estimate.

Table 2.1. Summatory psi functions and their first order asymptotic estimates

	Summatory psi			Summatory odd psi		
Limit	exact	appr.	err. %	exact	appr.	err. %
100	7'664	7'599	0.847	3'053	3'040	0.438
200	30'510	30'396	0.372	12'129	12'159	0.244
300	68'660	68'392	0.391	27'371	27'357	0.052
400	121'856	121'585	0.222	48'707	48'634	0.150
500	190'272	189'977	0.155	75'993	75'991	0.003
600	274'114	273'567	0.199	109'423	109'427	0.004
700	372'844	372'355	0.131	149'017	148'942	0.050
800	486'880	486'342	0.111	194'461	194'537	0.039
900	616'278	615'526	0.122	246'225	246'210	0.006
1000	760'410	759'909	0.066	303'873	303'964	0.030
2000	3'040'440	3'039'636	0.026	1'215'747	1'215'854	0.009
3000	6'841'542	6'839'180	0.035	2'735'127	2'735'672	0.020
4000	12'160'810	12'158'542	0.019	4'864'183	4'863'417	0.016
5000	19'001'312	18'997'722	0.019	7'598'249	7'599'089	0.011
6000	27'361'506	27'356'720	0.017	10'943'295	10'942'688	0.006
7000	37'240'964	37'235'535	0.015	14'894'381	14'894'214	0.001
8000	48'638'820	48'634'168	0.010	19'453'017	19'453'667	0.003
9000	61'560'598	61'552'619	0.013	24'621'307	24'621'048	0.001
10000	75'997'684	75'990'888	0.009	30'396'811	30'396'355	0.001

Table 2.1. (Continued)

Limit	Summatory even psi exact	appr.	err. %
100	4'611	4'559	1.118
200	18'381	18'238	0.779
300	41'289	41'035	0.615
400	73'149	72'951	0.270
500	114'279	113'986	0.256
600	164'691	164'140	0.334
700	223'827	223'413	0.185
800	292'419	291'805	0.210
900	370'053	369'316	0.199
1000	456'537	455'945	0.130
2000	1'824'693	1'823'781	0.050
3000	4'106'415	4'103'508	0.071
4000	7'296'627	7'295'125	0.021
5000	11'403'063	11'398'633	0.039
6000	16'418'211	16'414'032	0.025
7000	22'346'583	22'341'321	0.024
8000	29'185'803	29'180'501	0.018
9000	36'939'291	36'931'571	0.021
10000	45'600'873	45'594'533	0.014

3. Cumulative Number of Primitive Four Squares Representations

As an application of Proposition 2.1, we derive the asymptotic estimate (1.6) for the cumulative number of primitive representations of integers as sums of four squares. The exact value of the four squares primitive counting function is found in Cooper and Hirschhorn [2], Theorem 1, and reads

$$R_4(n) = \begin{cases} 8 \cdot \psi(s), & n = s, \\ 24 \cdot \psi(s), & n = 2s, \\ 16 \cdot \psi(s), & n = 4s, \\ 0, & 8 \mid n, \end{cases} \tag{3.1}$$

where the letter s denotes throughout an arbitrary odd number.

Theorem 3.1 (Cumulative number of primitive four squares representations)**.** *The summatory counting function* $N_4(x) = \sum_{n \le x} R_4(n)$ *satisfies the asymptotic relation*

$$N_4(x) = \frac{45}{\pi^2} \cdot x^2 + O(x \log x). \tag{3.2}$$

Proof. Using the formula (3.1), we partition the sum into three distinct parts such that (recall that s denotes an odd number)

$$N_4(x) = \sum_{n \le x} R_4(n) = 8 \cdot \sum_{s \le x} \psi(s) + 24 \cdot \sum_{2s \le x} \psi(s) + 16 \cdot \sum_{4s \le x} \psi(s)$$

$$= 8 \cdot \Psi^o(x) + 24 \cdot \Psi^o(x / 2) + 16 \cdot \Psi^o(x / 4),$$

which implies by Proposition 2.1 that

$$N_4(x) = \frac{3}{\pi^2} \{8 + 24 \cdot \frac{1}{4} + 16 \cdot \frac{1}{16}\} \cdot x^2 + O(x \log x) = \frac{45}{\pi^2} \cdot x^2 + O(x \log x).$$

The result is shown.

□

4. On the Asymptotic Cumulative Number of Primitive Pythagorean Quintuples

As explained in the Introduction, it is also of interest to consider the number of square representations by primitive sums of four squares, i.e., the solutions of

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 = n^2, \quad \gcd(x_1, x_2, x_3, x_4) = 1. \tag{4.1}$$

Formula (3.1) implies that

$$R_4(n^2) = \begin{cases} 8 \cdot \psi_2(s), & n = s, \\ 16 \cdot \psi_2(s), & n = 2s, \\ 0, & 4 \mid n, \end{cases} \tag{4.2}$$

where $\psi_2(s) = s\psi(s) = s^2 \cdot \prod_{p|s}(1 + \frac{1}{p})$. This formula shows that $NS_4(x) = \sum_{n \le x} R_4(n^2)$ depends on the odd summatory function $\Psi_2^o(x) = \sum_{\text{odd } n \le x} \psi_2(n)$. To obtain an asymptotic estimate for it, one considers similarly to Section 2 the related summatory functions $\psi_2(x) = \sum_{n \le x} \psi_2(n)$ and $\Psi_2^e(x) = \sum_{\text{even } n \le x} \psi_2(n)$.

Proposition 4.1. *The summatory second order Dedekind arithmetic functions satisfy the asymptotic relations*

$$\psi_2(x) = \frac{5}{\pi^2} \cdot x^3 + O(x^2 \log x),$$

$$\Psi_2^o(x) = \frac{2}{\pi^2} \cdot x^3 + O(x^2 \log x), \quad \Psi_2^e(x) = \frac{3}{\pi^2} \cdot x^3 + O(x^2 \log x). \tag{4.3}$$

Proof. With Overholt [11], Proposition 1.5, p. 11, one has

$$\psi_2(x) = \sum_{n \le x} n\psi(n) = x\Psi(x) - \int_1^x \Psi(u)du.$$

Inserting the estimate $\Psi(x)$ from Lemma 2.1 and integrating, one obtains the first formula in (4.3). Now, we proceed as in the proof of Proposition 2.1 and express the even $\Psi_2^e(x)$ as function of the odd $\Psi_2^o(x)$.

For even $2 \le n \le x$ there exists $1 \le k \le \lfloor \ln x / \ln 2 \rfloor$ such that $n = 2^k \cdot s$ with s odd and $1 \le s \le 2^{-k} \cdot x$. Using that $\psi_2(2^k s) = \frac{3}{2} 2^{2k} \psi_2(s)$, one sees that

$$\Psi_2^e(x) = \sum_{k=1}^{\lfloor \ln x / \ln 2 \rfloor} \left\{ \sum_{\text{odd } s \le 2^{-k} x} \psi_2(2^k s) \right\} = \frac{3}{2} \cdot \sum_{k=1}^{\lfloor \ln x / \ln 2 \rfloor} 2^{2k} \cdot \Psi_2^o(2^{-k} \cdot x).$$

Now, suppose that $\Psi_2^o(x) = cx^3 + O(x^2 \log x)$ for some constant c. Using that $\Psi_2(x) = \Psi_2^o(x) + \Psi_2^e(x)$, the first formula in (4.3), and the preceding identity, one must have

$$\frac{5}{\pi^2} \cdot x^2 \sim cx^3 + \frac{3}{2} \left(\sum_{k=1}^{\lfloor \ln x / \ln 2 \rfloor} 2^{-k} \right) cx^3 \quad (x \to \infty),$$

which holds exactly when $\frac{5}{2} c = \frac{5}{\pi^2}$, or $c = \frac{2}{\pi^2}$. The result follows. □

Table 4.1 compares the exact values with their asymptotic estimates for $x \le 10^4$. Up to the slightly increased percentage relative error, the same comments as for Table 2.1 hold.

Table 4.1. Summatory second order psi functions and their first order asymptotic estimates

	Summatory psi2			Summatory odd psi2		
Limit	exact	appr.	err. %	exact	appr.	err. %
100	514'631	506'606	1.559	203'981	202'642	0.656
200	4'081'711	4'052'847	0.707	1'615'225	1'621'139	0.366
300	13'772'625	13'678'360	0.684	5'475'579	5'471'344	0.077
400	32'555'513	32'422'779	0.408	12'998'219	12'969'112	0.224
500	63'510'955	63'325'740	0.292	25'331'161	25'330'296	0.003
600	109'809'897	109'426'878	0.349	43'768'239	43'770'751	0.006
700	174'182'531	173'765'830	0.239	69'558'737	69'506'332	0.075
800	259'910'575	259'382'230	0.203	103'692'085	103'752'892	0.059
900	370'116'183	369'315'714	0.216	147'739'545	147'726'286	0.009
1000	507'258'671	506'605'918	0.129	202'552'529	202'642'367	0.044
2000	4'055'065'425	4'052'847'346	0.055	1'620'925'683	1'621'138'938	0.013
3000	13'686'815'641	13'678'359'792	0.062	5'469'712'927	5'471'343'917	0.030
4000	32'434'287'553	32'422'778'766	0.035	12'972'174'487	12'969'111'506	0.024
5000	63'347'487'531	63'325'739'776	0.034	25'326'096'417	25'330'295'911	0.017
6000	109'461'077'409	109'426'878'334	0.031	43'774'388'991	43'770'751'333	0.008
7000	173'811'281'783	173'765'829'947	0.026	69'507'500'021	69'506'331'979	0.002
8000	259'429'185'395	259'382'230'124	0.018	103'747'686'209	103'752'892'050	0.005
9000	369'399'842'229	369'315'714'376	0.023	147'728'611'203	147'726'285'751	0.002
10000	506'689'079'217	506'605'918'212	0.016	202'646'936'259	202'642'367'285	0.002

Table 4.1. (Continued)

Limit	Summatory even psi2 exact	appr.	err. %
100	310'650	303'964	2.152
200	2'466'486	2'431'708	1.410
300	8'297'046	8'207'016	1.085
400	19'557'294	19'453'667	0.530
500	38'179'794	37'995'444	0.483
600	66'041'658	65'656'127	0.584
700	104'623'794	104'259'498	0.348
800	156'218'490	155'629'338	0.377
900	222'376'638	221'589'429	0.354
1000	304'706'142	303'963'551	0.244
2000	2'434'139'742	2'431'708'407	0.100
3000	8'217'102'714	8'207'015'875	0.123
4000	19'462'113'066	19'453'667'259	0.043
5000	38'021'391'114	37'995'443'866	0.068
6000	65'686'688'418	65'656'127'000	0.047
7000	104'303'781'762	104'259'497'968	0.042
8000	155'681'499'186	155'629'338'075	0.034
9000	221'671'231'026	221'589'428'626	0.037
10000	304'042'142'958	303'963'550'927	0.026

We are ready for our second main result.

Theorem 4.1 (Cumulative number of square representations by primitive sums of four squares)**.** *The summatory counting function* $NS_4(x) = \sum_{n \le x} R_4(n^2)$ *satisfies the asymptotic relation*

$$NS_4(x) = \frac{20}{\pi^2} \cdot x^3 + O(x^2 \log x). \tag{4.4}$$

Proof. With (4.2) it suffices to partition the sum into two distinct parts such that

$$NS_4(x) = \sum_{n \le x} R_4(n^2) = 8 \cdot \sum_{s \le x} \psi_2(s) + 16 \cdot \sum_{2s \le x} \psi_2(s) = 8 \cdot \Psi^o(x) + 16 \cdot \Psi^o(x/2),$$

which implies that $NS_4(x) = \frac{2}{\pi^2}\{8 + 16 \cdot \frac{1}{8}\} \cdot x^3 + O(x^2 \log x) = \frac{20}{\pi^2} \cdot x^3$ $+ O(x^2 \log x)$. □

It is interesting to compare (4.4) with the cumulative number of distinct primitive Pythagorean quintuples without zeros, for which Hürlimann [5] obtains an exact formula. While (4.4) takes into account possible zeros, permutations and sign changes, the exact formula considers only the distinct ones without zeros. If, for a meaningful comparison, one adjusts for the permutations and sign changes, then a distinct primitive Pythagorean quintuple without zeros generates at most 384 primitive representations within the present counting mechanism. Heuristically, one can compare the cumulative number of distinct primitive Pythagorean quintuples without zeros with the principal component

$$P_4(x) = NS_4(x)/384 = \frac{5}{96\pi^2} \cdot x^3 + O(x^2 \log x) \qquad (4.5)$$

obtained from (4.4). Table 4.2 below illustrates.

Table 4.2. Number of primitive Pythagorean quintuples without zeros versus formula (4.5)

	Principal component with zeros			Primitive Pythagorean without zeros	
Limit	exact	asymptotic	err. %	exact	ratio in %
100	5'295	5'277	0.332	5'568	95.09
200	42'150	42'217	0.160	43'238	97.48
300	142'605	142'483	0.085	145'040	98.32
400	338'097	337'737	0.106	342'421	98.74
500	659'842	659'643	0.030	666'610	98.98
600	1'139'987	1'139'863	0.011	1'149'715	99.15
700	1'810'379	1'810'061	0.018	1'823'605	99.27
800	2'701'844	2'701'898	0.002	2'719'144	99.36
900	3'846'415	3'847'039	0.016	3'868'296	99.43
1000	5'275'309	5'277'145	0.035	5'302'303	99.49

It is remarkable that the principal component (4.5) approaches with increasing limit the exact cumulative number of distinct primitive Pythagorean quintuples from Table 5.2 in Hürlimann [5]. Based on this, we conjecture that the principal component yields asymptotically the correct first order estimate. Also, the extension of the obtained results to higher dimensions might be an interesting topic for future research.

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