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THE CRITERIA OF CHECKING HYPOTHESES OF QUANTUM STATES OF QUANTUM PHYSICAL SYSTEMS

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Abstract

In this article, we define and build the criteria for the testing of hypotheses in the physical quantum systems.

For each quantum state, we construct the measure in the space (P_U, S), where P_U is the space of the pure states of the quantum system, S is the Borel σ-algebra in P_U. The criteria for the checking hypotheses of the quantum states we call a measurable map $\delta : (P_U, S) \to (2^{\Im}, \Omega)$, where $\Im$ is the space of the hypothetical states with the topological structure induced by topological structure in the space quantum states $\Im_U$, $2^{\Im}$ is the space of closed subsets of $\Im$ with the exponential topological structure, and Ω is the Borel σ-algebra subsets in $2^{\Im}$.

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1. Quantum Physical Systems

As known [1] a quantum physical system can represented by a couple $(U, \Im)$, where U is some C^*-algebra which Hermit's the elements that are called observables and some set $\Im$ of positive functionals with norm one and called the quantum states of this physical system [2]. We say what the functional ω_1 majorizes functional ω_2 if $\omega_1 - \omega_2$ is a positive functional. The state ω quantum physical system is called the pure state if it majorizes only functional type $\lambda\omega$, $0 \leq \lambda \leq 1$ [2].

Denote by P_U the set of pure states on C^*-algebra U.

In the set of all linear continue functional on C^*-algebra U, we have topological structure which called as weakly topological structure [2] and which defined by pre basis:

$$V(\omega, u_1, u_2, \ldots, u_n) = \{\omega' \in U^* | \ |\omega(u_i) - \omega'(u_i)| < \varepsilon, \quad i = 1, 2, \ldots, n\},$$

where $\omega, \omega' \in U^*$, $u_i \in U$, accordingly this in the set P_U, we have the topological structure induced from this topological structure.

Denote by $\mathfrak{R}$ the set of Hermit's elements of U C^*-algebra.

Lemma. *Every linear functional on the U C^*-algebra uniquely will be defined by its values on Banach subspace of Hermit's elements.*

Proof. It's known [2] that every element u of C^*-algebra U uniquely represented as $u_1 + iu_2$, where u_1 and u_2 are Hermitian. The lemma is proved.

Every a Hermit's element $u \in \mathfrak{R}$ in the C^*-algebra U any can be represented by integral

$$u = \int_{-\infty}^{\infty} \lambda dp_\lambda^u,$$

where $\{p_\lambda^u\}_{\lambda \in R} (p_\lambda^u)^2 = p_\lambda^u$ projectors and represents the partition of unity of Hermit's element $u \in \mathfrak{R}$ [4].

Correspond to projector $p_\alpha^2 = p_\alpha \in U$ the family $\{p_\lambda^{p_\alpha}\}_{\lambda \in R}$ of elements of the C^*-algebra U which has the condition: $p_\lambda^{p_\alpha} = 0$, if $\lambda < 1$, $p_\lambda^{p_\alpha} = p_\alpha$, if $\lambda = 1$ and $p_\lambda^{p_\alpha} = E$, if $\lambda > 1$, where E is the unit element in the algebra U. It is clear that

$$p_\alpha = \int_{-\infty}^{\infty} \lambda dp_\lambda^{p_\alpha}.$$

If $u \in U$, then by the proved lemma $u = u_1 + iu_2$, where u_1 and u_2 are Hermitian elements. The representation such u will be

$$u = u_1 + iu_2 = \int_{-\infty}^{\infty} \lambda dp_\lambda^{u_1} + i\int_{-\infty}^{\infty} \lambda dp_\lambda^{u_1}.$$

Obviously, if ω is some linear continue functional on C^*-algebra U, then from lemma and from the last equality we will have

$$\omega(u) = \omega(u_1 + iu_2) = \omega(\int_{-\infty}^{\infty} \lambda dp_\lambda^{u_1}) + \omega(\int_{-\infty}^{\infty} \lambda dp_\lambda^{u_2}) = \int_{-\infty}^{\infty} \lambda d\omega(p_\lambda^{u_1}) + i\int_{-\infty}^{\infty} \lambda d\omega(p_\lambda^{u_2}),$$

where u_1 and u_2 are Hermit's elements.

Let $\{p_\alpha\}$ be the set of all one dimensional projectors on C^*-algebra U and ω be pure state, then from lemma and the last equality follows that this state has the non zero meaning only on some projector $p_\alpha \in \{p_\alpha\}$ and the meaning 0 on the other one dimensional projectors. Otherwise, we can always construct such functional which will not have the type $\lambda\omega$, $0 \le \lambda \le 1$ and well majorized by the pure state ω. So how, if functional ω is a pure state, then $\omega(u^2) \ge 0$ for all Hermit's elements u. Let $\omega(p_{\alpha'}) \ne 0$ and $\omega_{\alpha'}$ such functional, which has the non zero meaning ε only on some projector $p_{\alpha'} \in \{p_\alpha\}$ and the meaning 0 on the other one dimensional projectors. It is clear, if we take ε sufficiently small, then we can achieve, that for every $u \in \mathfrak{R}$ will have place inequality

$$(\omega - \omega_{\alpha'})(u^2) = \int_{-\infty}^{\infty} \lambda^2 d(\omega - \omega_{\alpha'})(p_\lambda^u) \geq 0.$$

It means, that the pure state ω majorize the functional $\omega_{\alpha'}$ which does not have the type $\lambda\omega$, $0 \leq \lambda \leq 1$, but this is impossible. It follows that the pure states are such functional which satisfy the condition $\omega_\alpha(p_\beta) = \delta^{\alpha\beta}$.

An integral representation of Hermit's elements follows that for the pure states ω_α have place the equality $\omega_\alpha(u) = \lambda_\alpha^u$, where λ_α^u is some element of spectrum of Hermit's element $u \in \mathfrak{R}$. It gives opportunity identify every pre state with the set of number $\{\lambda_\alpha^u\}_{u \in \mathfrak{R}}$, where $p_\alpha(u) = \lambda_\alpha^u$. Consider the Tikhonov's product $\Sigma = \underset{u \in \mathfrak{R}}{\otimes} \sigma_u$, where $\sigma_u \subset R$ spectrum of element $u \in \mathfrak{R}$.

It is clear, $P_U \subset \Sigma$, because P_U the set of such elements in product $\Sigma = \underset{u \in \mathfrak{R}}{\otimes} \sigma_u$ which represents linear continue maps with respect to the topological structure in $\mathfrak{R}$, which is defined by the norm:

$$P_U = \{\omega_\alpha : \mathfrak{R} \to \underset{u \in \mathfrak{R}}{\cup} \sigma_u \mid \omega_\alpha(u) = \lambda_\alpha^u,\ \omega_\alpha(k_1 u_1 + k_2 u_2) = k_1 \omega_\alpha(u_1) + k_2 \omega_\alpha(u_2)\}.$$

Consequently, in the set P_U, we have $\Sigma = \underset{u \in \mathfrak{R}}{\otimes} \sigma_u$, induced from Tikhonov's product $\Sigma = \underset{u \in \mathfrak{R}}{\otimes} \sigma_u$ topological structure. Lemma follows, that this topological structure coincides with the induced topological structure from weakly topological structure on set of functionals on C^* algebra U. We can also identify the set P_U with the set of one dimensional projectors $\{p_\alpha\}$.

Theorem 1. *Every state* $\omega \in \Im$ *in space* P_U *with weakly topological structure, defined on the Borel* σ*-algebra of a probability measure* μ_ω.

Proof. Let $\omega \in \Im$ be some state. Consider the positive functionals ν on C^*-algebra whose value is non zero only on one dimensional projectors p_α for which $\omega(p_\alpha) \neq 0$ and in addition $\nu(p_\alpha) = \omega(p_\alpha)$. If such functional exists, then it is clear that $\|\nu\| \leq \|\omega\| \leq 1$.

Let new ν_1 and ν_2 such functional and in addition if $\nu_1(p_\alpha) \neq 0$, then $\nu_2(p_\alpha) = 0$. As known [2] for the positive functional has place the equality $\|\nu_1\| + \|\nu_2\| = \|\nu_1 + \nu_2\|$, for our case it follows $\|\nu_1 + \nu_2\| \leq 1$. Consider the subset $\{p_{\alpha_\beta}\}$ of P_U, such positive functional ν, $\|\nu\| \leq 1$, if this exists, which values on the elements of this subset are coincide to corresponding values of the state ω and the relevance $\mu_\omega : \{\{p_{\alpha_\beta}\}\} \to R$, which defined by the equality $\mu_\omega(\{p_{\alpha_\beta}\}) = \|\nu\|$. This relevance can not be identified for every subset in P_U, but it is σ-additive. Indeed, for functionals ν_1 and ν_2 whose satisfied the condition: if $\nu_1(p_\alpha) \neq 0$, then $\nu_2(p_\alpha) = 0$, has place equality $\|\nu_1\| + \|\nu_2\| = \|\nu_1 + \nu_2\|$.

For finite sum $\nu_1 + \nu_2 + \ldots + \nu_n$, where every ν_i such that if $\nu_i(p_\alpha) \neq 0$, then $\nu_j(\rho_\alpha) = 0$ when $i \neq j$ also has place the equality

$$\|\nu_1\| + \|\nu_2\| + \ldots + \|\nu_n\| = \|\nu_1 + \nu_2 + \ldots + \nu_n\| \leq 1.$$

Let the given series of positive functionals $\nu_1 + \nu_2 + \ldots + \nu_n + \ldots$, where every ν_i such that if $\nu_i(p_\alpha) \neq 0$ then $\nu_j(\rho_\alpha) = 0$ when $i \neq j$. This series is converge, otherwise the norm of the functional $\omega \in \Im$ will not be one.

The sum of this series is $\lim\limits_{n \to \infty}(\nu_1 + \nu_2 + \ldots + \nu_n)$.

$$\lim_{n \to \infty} \|\nu_1 + \nu_2 + \ldots + \nu_n\| = \lim_{n \to \infty}(\|\nu_1\| + \|\nu_2\| + \ldots + \|\nu_n\|) \leq 1.$$

If limit include in norm symbol we obtain equality

$$\| \lim_{n\to\infty} (\nu_1 + \nu_2 + \ldots + \nu_n)\| = \lim_{n\to\infty} (\|\nu_1\| + \|\nu_2\| + \ldots + \|\nu_n\|).$$

It follows that

$$\|\nu_1\| + \|\nu_2\| + \ldots + \|\nu_n\| + \ldots = \|\nu_1 + \nu_2 + \ldots + \nu_n + \ldots\| \le 1,$$

It means that relevance $\mu_\omega : \{\{p_{\alpha_\beta}\}\} \to R$ is σ-additive.

If $\omega \in \Im$ is pure state then the according measure will be the Dirac's measure [2]. In space P_U with weakly topological structure the closure of set such points whose elements represents all projectors on which state $\omega \in \Im$ are different from zero is called as support of this state. It is denoted so: *Support*ω.

Let $\{p_{\alpha_\beta}\}$ be the closed subset in space P_U with weakly topological structure. Consider set $\{p_{\alpha_\beta}\} \cap Support\omega$. The functional ν, which coincide to $\omega \in \Im$ on this intersection and is zero on the other one dimensional projectors will be continue, it follows that ν is positive [2].

It is clear that $\|\nu\| \le 1$. It means that the relevance $\mu_\omega : \{\{p_{\alpha_\beta}\}\} \to R$, $\mu_\omega(\{p_{\alpha_\beta}\}) = \|\nu\|$ is defined for all closed subsets of P_U with weakly topological structure.

If $\{p_{\alpha_\beta}\}$ is the closed subset in P_U consider $\overline{P_U \setminus \{p_{\alpha_\beta}\} \cap Support\omega}$, there exists such ν, $\|\nu\| \le 1$ positive functional that $\mu_\omega(\{p_{\alpha_\beta}\}) = \|\nu\|$. Let $\overline{\nu}$ be such functional which coincide to $\omega \in \Im$ functional on this intersection and is zero on other one dimensional projectors. This functional is continue and therefore positive. It is clear that $\|\overline{\nu}\| = 1 - \|\nu\|$. We will assume $\mu_\omega(\{P_U \setminus \{p_{\alpha_\beta}\}\}) = \|\overline{\nu}\|$. It follows that the relevance $\mu_\omega : \{\{p_{\alpha_\beta}\}\} \to R$, $\mu_\omega(\{p_{\alpha_\beta}\}) = \|\nu\|$ is also defined for open subsets of P_U with weakly topology. So ultimately we built the probability measure on the Borel σ-algebra of space P_U with weakly topological structure.

The theorem is proved.

We call space P_U with weakly topological, as space of quantum physical system. If C^*-algebra U has a unit, then in the space U^* with weakly topological structure the set of al state $\Im$ is convex compact set and represent convex linear combination of pure states $\zeta_1, \zeta_2, \ldots, \zeta_n$ from the set P_U :

$$\omega = k_1\zeta_1 + k_2\zeta_2 + \ldots + k_n\zeta_n,\ k_i \geq 0,\ \sum_{i=1}^{n} k_i = 1,$$

or limit of sequence $\omega_1, \omega_2, \ldots, \omega_l$, where $\omega_l = k_1^l\zeta_1^l + k_2^l\zeta_2^l + \ldots + k_{n_l}^l\zeta_{n_l}^l$, $k_i^l \geq 0, \sum_{i=1}^{n_l} k_i^l = 1$ [2]. This means, that elements of set P_U are the extreme points of set [2].

Because each state $\omega \in \Im_u$ define sprobabiliti measure μ_ω on couple (P_U, S), where S is borel C^*-algebra therefore it ease to show, that every μ_ω represent convex linear combination

$$\mu_\omega = \sum_{i=1}^{n} k_i\mu_{\zeta_i},\ k_i \geq 0,\ \sum_{i=1}^{n} k_i = 1,$$

of Dirak measures μ_{ζ_i} where $\xi_i \in P_U$, $i = 1, 2, \ldots, n$ or limit of sequense $\{\mu_{\omega_n}\}_{n\in N}$, where $\mu_{\omega_n} = k_1^n\mu_{\zeta_1^n} + k_2^n\mu_{\zeta_2^n} + \ldots + k_{n_{ll}}^n\mu_{\zeta_l^n}$, $k_l^n \geq 0$, $\sum_{i=1}^{l} k_i^n = 1$ [2].

For every state $\omega \in \Im$, we have $\int_{-\infty}^{\infty} d\omega(E_\lambda^u) = 1$, therefore it is easy that the value of quantum state on observable $u \in \Re$ is the middle value of this observable. The value $\omega(u) \in R$ is called the middle value of observable $u \in \Re$ of quantum physical system in the state $\omega \in \Im$.

All told above follows that a quantum physical system is object

$$(U,\ P_U,\ S,\ \mu_\omega,\ \omega \in \Im),$$

where U is some C^*-algebra, Hermit element of which are called observables of this system, P_U is the space of quantum system, S is Borel σ-algebra in P_U, and μ_ω is the probability measure defined by state $\omega \in \Im$.

2. Criteria of Checking Hypotheses of Quantum States of Quantum Physical Systems

The space of quantum system is location of quantum particles therefore the supports of states can $\omega \in \Im$ covers this space that is $\bigcup_{\omega \in \Im} Supp\omega = P_U$.

Let given a quantum physical system

$$(U,\ P_U,\ S,\ \mu_\omega,\ \omega \in \Im),$$

and Ω the Borel's σ-algebra in the space $2^\Im$ of closed subsets in $\Im$ with exponential topological structure, which is induced from weakly topological structure.

Definition 1. We say, that the quantum physical system $(U,\ P_U,\ S,\ \mu_\omega,\ \omega \in \Im)$ admits criteria for checking hypotheses of finding in states $\omega \in \Im$ with probability $p \geq \alpha,\ 0 \leq \alpha \leq 1,$ if there exists measurable map

$$\delta : (P_U,\ S) \to (2^\Im,\ \Omega),$$

which satisfied condition

$$\mu_\omega\{x \in P_U \mid \delta(x) = \omega \in \Im \subset 2^\Im\} = p \geq \alpha,$$

for all state $\omega \in \Im$.

Definition 2. The value $p_\omega(\delta) = \mu_\omega\{x \in P_U \mid \delta(x) \neq \omega\}$ is called the probability of ω-kind of mistake for δ criteria.

It is easy, that a map $\delta : (P_U, S) \to (2^{\Im}, \Omega)$ then, and only then represents consistent criteria for checking hypotheses of finding in the state $\omega \in \Im$ with the probability $p \geq \alpha, 0 \leq \alpha \leq 1$ when the value $p_\omega(\delta) = \mu_\omega\{x \in P_U \mid \delta(x) \neq \omega\} = q < 1 - \alpha$ for all states $\omega \in \Im$.

Theorem 2. *Let $\Im_x$ be the closure in $\Im$ of set of all such states $\omega \in \Im$ which supports contains the point $x \in P_U$, then the map*

$$\delta : (P_U, S) \to (2^{\Im}, \Omega),$$

where $\delta(x) = \Im_x$, is continue.

Proof. The basis in space $2^{\Im}$ is consists of sets:

$$V(O_1, O_2, \dots, O_k) = \{B \in 2^{\Im} \mid B \subset \bigcup_{i=1}^{k} O_i;\ B \cap O_i \neq \varnothing, i = 1, 2, \dots, k\},$$

where $O_1, O_2, \dots, O_k$ is open sets in space $\Im$ with topological structure which induced from weakly topological structure. Let W be an open set in the space $2^{\Im}$. We will show that the set $\delta^{-1}(W)$ is also open. The sets $W = \bigcup_\beta V(O_1^\alpha, O_2^\beta, \dots, O_{k_\beta}^\beta)$, where

$$V(O_1^\beta, O_1^\beta, \dots, O_{k_\beta}^\phi) = \{B \in 2^{\Im} \mid B \subset \bigcup_{i=1}^{k_\beta} O_i^\beta; B \cap O_i^\beta \neq \varnothing, i = 1,2,\dots, k_\beta\}.$$

Supports of $\omega \in \Im$ cover P_U, therefore $\delta^{-1}(W) = \bigcup_\beta \bigcup_{i=1}^{k_\beta} \bigcup_{\omega_i^\beta \in O_i^\beta} \mathit{Support}\ \ \omega_i^\beta$.

The set $\bigcup_\beta \bigcup_{i=1}^{k_\beta} O_i^\beta$ is opened. If $\delta^{-1}(W)$ is not opened there exists pure state $\omega_\alpha \in Fr\delta^{-1}(W)$. If $\omega_\alpha \in \mathit{Support}\,\omega$, where $\omega \in \bigcup_\beta \bigcup_{i=1}^{k_\beta} O_i^\beta$, then

$\omega \in Fr\bigcup_{\beta}\bigcup_{i=1}^{k_\beta} O_i^\beta$, but $Fr\bigcup_{\beta}\bigcup_{i=1}^{k_\beta} O_i^\beta, = \varnothing$, it follows, that we have the contradiction, our assumption of existence $\omega_\alpha \in Fr\delta^{-1}(W)$ is false. It follows that the set $\delta^{-1}(W)$ is opened and δ continue. The theorem is proved.

Each continue map is measurable, therefore if for δ

$$\mu_\omega\{x \in P_U \mid \delta(x) = \omega \in \Im \subset 2^{\Im}\} = p \geq \alpha,$$

then quantum physical system admits criteria for checking hypotheses of finding in states $\omega \in \Im$ with probability $p \geq \alpha$.

Definition 3. A quantum physical system is called orthogonal if for each two quantum state $\omega, \omega' \in \Im, \omega \neq \omega'$ corresponding measures μ_ω, $\mu_{\omega'}$ are orthogonal, i.e., there exists subset $N_{\omega'} \subset P_U$ such that $\mu_\omega(N_{\omega'}) = 1$ and $\mu_{\omega'}(P_U \setminus N_{\omega'}) = 0$.

Definition 4. Quantum physical system $(U, P_U, S, \mu_\omega, \omega \in \Im)$ is called the strongly separable quantum physical system if there exist disjoint family S-measurable sets $\{X_\omega, \omega \in \Im\}$ such that the relations are fulfilled: $\forall\omega \in \Im\, \mu_\omega(X_\omega) = 1$.

Remark. It's known that every strongly separable quantum physical system are orthogonal [3].

Definition 5. We say, that the quantum physical system $(U, P_U, S, \mu_\omega, \omega \in \Im)$ admits consistent criteria for checking hypotheses of finding in states $\omega \in \Im$ with probability $p \geq \alpha, 0 \leq \alpha \leq 1$, if there exists measurable map

$$\delta : (P_U, S) \to (2^{\Im}, \Omega),$$

which satisfied condition

$$\mu_\omega\{x \in P_U \mid \delta(x) = \omega \in \Im \subset 2^{\Im}\} = 1,$$

for all state $\omega \in \Im$.

Theorem 3. *If quantum physical system* $(U, P_U, S, \mu_\omega, \omega \in \Im)$ *is strongly separable, then it admits consistent criteria for checking hypotheses of finding in states* $\omega \in \Im$.

Proof. The Theorem 2 tells us that there exists continue map

$$\delta : (P_U, S) \to (2^{\Im}, \Omega),$$

which defined by formula $\delta(x) = \Im_x$. For our case $\mu_\omega\{x \in P_U \mid \delta(x) = \omega \in \Im \subset 2^{\Im}\} = 1$.

The theorem is proved.

The Theorem 3 and remark, which are made above, follow, that every orthogonal quantum physical system admits consistent criteria for hypotheses.

References

[1] H. H. Bogolyubov, A. A. Logunov, A. I. Oksak and I. T. Todorov, General Principles of the Quantum Field Theories, Moscow, Nauka, 1987 (in Russian).

[2] U. Bratteli and D. Robinson, Operator Algebras and Quantum Statistical Mechanics, Moscow, Nauka, 1982 (in Russian).

[3] Z. Zerakidze and M Mumladze, Statistical Structures and Consistent Criteria for Checking Hypotheses, LAMBERT Academic Publishing, 2015.

[4] L. A. Liusternirnik and V. I. Sobolev, Elements of Functional Analyses, Moscow, Nauka, 1965 (in Russian).

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