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THE RIEMANN HYPOTHESIS ABOUT THE NON-TRIVIAL ZEROES OF THE ZETA FUNCTION

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Abstract

The aim of the present essay is to investigate whether it is possible to approach the age-old problem of the hypothesis by elementary algebraic means. However, the results about the properties of the Zeta function acquired by professional mathematicians in more than 150 years of research using advanced theories and methods are here taken for granted and freely taken avail of. The necessary implication of our developments appears to be that non-trivial zero-points cannot exist outside the critical line $X = \frac{1}{2}$.

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1. Basic Properties of the Zeta Function

According to Hardy [1], the non-trivial zeroes of $Z(s)$, consisting in complex conjugate (c.c.) pairs of zero-points, are infinite in number and must be contained within the critical strip (C.S.) $0 < X < 1$, where $X = \text{Re}(s)$ and $s = X + i.Y$. The original definition of the Zeta function, $Z(s) = \sum_{n=1}^{\infty} n^{-s}$, does not lend itself to easy elaborations, because such a series is convergent only in the half-plane $X > 1$. Therefore, it is necessary to find extensions of this definition that give the same results as the original formula for $X > 1$, but converge also within the C.S.

We chose to work with the extension by Gram and Backlund [2], as reported by Edwards [3] (in particular, see Subsection 6.4, pages 114 to 118 which contain also a detailed discussion of the convergence of the series involved):

$$Z(s) = \sum_{n=1}^{N-1} n^{-s} + \frac{N^{1-s}}{s-1} + \frac{1}{2}.N^{-s} + \frac{B_2}{2}.s.N^{-s-1}$$

$$+ \dots + \frac{B_{2\nu}}{(2.\nu)!}.s.(s+1).(s+2)\dots(s+2.\nu-2).\ N^{-s-2.\nu+1} + R_{2\nu}, \qquad (1)$$

where the term $R_{2\nu}$ is a remainder that can be made as small as necessary by a judicious choice of the large integer N and of ν, and the coefficients $B_{2\nu}$ are Bernoulli numbers. The Riemann hypothesis states that the c.c. pairs of non-trivial zeroes lie exclusively on the critical line $X = \frac{1}{2}$ (C.L.), and this was verified up to extremely high values of Y. But this gives no certainty that the hypothesis can be verified in general. An important feature of the distribution of zeroes of $Z(s)$ was discovered by Hadamard [4] and De La Vallee-Poussin (H-DLVP) [5], who showed that if 'outlying' zero-points of $Z(s)$, i.e., zero-points not belonging to the C.L. should exist, they must occur as two pairs of c.c. zeroes, one the mirror-image of the other by reflection with respect to the C.L.

2. Assumptions Made in the Present Essay

The above-cited established results, i.e., the Gram-Backlund expression of the extension of $Z(s)$ and the H-DLVP theorem concerning the outlying zeroes, if any, are the properties used in our essay and their assumption is our starting point, essential for the kind of argumentation we intend to develop. The Gram-Backlund expression of the zero-condition for $Z(s)$, according to Equation (1), is

$$Z(s) = \sum_{n=1}^{N-1} n^{-s} + \frac{N^{1-s}}{s-1} + \frac{1}{2}.N^{-s} + \frac{B_2}{2}.s.N^{-s-1}$$

$$+ \dots + \frac{B_{2\nu}}{(2.\nu)!}.s.(s+1).(s+2)\dots(s+2.\nu-2).\ N^{-s-2.\nu+1} + R_{2\nu} = 0. \quad (2)$$

Equation (2) is transformed, taking account of the fact that the non-trivial zero-points include neither $s = 0$ nor $s = 1$, into the following form[1]:

$$Z(s) = A(s).\left[\frac{1}{s-1} + \frac{s}{Q(s)}\right] = 0, \quad (3)$$

where

$$A(s) = N^{1-s} = \exp[(1-s).\ln N],$$

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[1]The algebraic form (3) represents in compact notation the zero-condition (2) which derives strictly from the Euler-McLaurin summation by integration of the part $\sum_{n=N}^{\infty} n^{-s}$ of the sum $\sum_{n=1}^{\infty} n^{-s}$. The Euler-McLaurin summation is used in the evaluation of $Z(s)$ by a considerable number of classical, as well as modern, students of the subject, and this, as shown, e.g., in the Gram-Backlund papers, entails the presence of other terms containing the factor s. Therefore, the algebraic form (3), or the equivalent form (7) can be taken as representative in general.

We show (Sections 3 and 4) that this form makes any hypothetical ‘outlying’ c.c. pair incompatible with Equations (13) and (14), while a c.c. zero-pair lying on the C.L. is compatible, see Equations (15) to (20).

and

$$\frac{1}{Q(s)} = \left\{ \sum_{\mu=2}^{\nu} \frac{B_{2\mu}}{(2.\mu)!} .N^{-2.\mu}.(s+1).(s+2)\dots(s+2.\nu-2) \right.$$

$$\left. + \frac{1}{s}\left[\frac{1}{N}\left(\sum_{n=1}^{N-1}\left(\frac{n}{N}\right)^{-s}\right) + \frac{1}{2} + N^{s}.R_{2\nu}\right]\right\}. \qquad (4)$$

Equation (4) can be easily obtained from (2) by legitimate algebraic operations.

At last, we assume that an instance of outlying zero-points after H-DLVP has been found, so that the zero-condition (3) is verified in any of the four H-DLVP zeroes, and we proceed to derive the implications of the whole set of our assumptions.

But in order to better follow the logical framework of our discussion, it is advisable to evidence which are, in our opinion, the original aspects of our essay.

Past attempts at proving or disproving the Riemann hypothesis were based on one or the other of two different approaches:

Search for a proof. These attempts start generally from the known properties of the Zeta function and try to use the tools of higher mathematics. So far, the attempts based on this choice -undoubtedly an undisputable one- have been unfruitful.

Search for a disproof. These attempts were generally based on the 'brute-force' approach, i.e., on numerical exploration of the C.S. in order to detect the outlying zero-points, if any. As such, they necessarily use 'valid' extensions of the Zeta function. These extensions are as well necessarily used in the determination of the canonical zeroes (i.e., those lying on the C.L.). A theoretical discussion about the zeroes based on a 'valid' extension of the Zeta function could not, in our opinion, be objected to for its 'hybrid' nature[2]. Indeed, to cast doubts about the reliability of

[2]By 'valid', we intend an extension recognized as a legitimate one; in particular, we term as 'valid' any extension which was used to compute the canonical zeroes.

the conclusions of such an approach about the occurrence of outlying zeroes would implicitly allow to question also the reliability of recognized canonical zeroes.

Such a 'hybrid' approach was, indeed, the choice we selected as our basic ground. The hopes we entertained on the effectiveness of such a choice can be summed up as follows.

The extension we chose to work with, i.e., the Gram-Backlund extension, conceptually replaces the zero-condition for the Zeta function, $Z(s)$, with the zero-condition for or a directly related function $z^*_{GB}(s)$ having the same zeroes. Our function $z^*_{GB}(s)$ consists of the sum of two functions, say $z^*_1(s) + z^*_2(s)$, of which $z^*_1(s)$ is an algebraic function while $z^*_2(s)$ is a transcendental analytical function. Then we assume that an outlying zero has been found at $s = s_0$, so that the local value of $z^*_2(s)$ has to be regarded as a known quantity $Q(s_0)$ (assumed for generality to be a complex quantity). We can thus analyze the compatibility of the known properties of algebraic functions, applied to $z^*_1(s)$, with $z^*_1(s) + Q(s_0) = 0$ and with other conditions (such as the Hadamard-De La Vallee Poussin theorem about the existence of two pairs of outlying zeroes specularly disposed on the two sides of the C.L.) known from past studies to be applicable to $z^*_1(s) + z^*_2(s)$. We can analyze as well the compatibility of the known properties of analytical functions, applied to $Q(s)$, with $z^*_1(s_0) + Q(s) = 0$. This double check appears, in our analysis, to have synergic effects capable of yielding meaningful results.

3. Necessary Implications of the Gram-Backlund Zero-Condition and of the H-DLVP Theorem

We begin by noting that in (3) it is

$$A(s) \neq 0, \frac{1}{Q(s)} \neq 0, \quad Q(s) \neq 0. \tag{5}$$

Indeed, since N is a large positive integer and $s \neq 1$, $\operatorname{Re}(1 - s) > 0$, it follows:

$$\exp[(1 - s). \ln N] \neq 0. \tag{6}$$

Nor can it be $\frac{1}{Q(s)} = 0$, because in that case the zero-condition (3) would require $\frac{1}{s - 1} = 0$, a meaningless statement in the frame of the quest for outlying zero-points. And likewise the case $Q(s) = 0$ can be excluded, because then the zero-condition (3) would require $s = 1$, where $Z(s)$ has a pole instead of a zero, as abundantly known.

On the strength of (5), the synthetic form (3) of the zero-condition can be turned into the form

$$s.(s - 1) + Q(s) = 0 \text{ or } s^2 - s + Q(s) = 0, \tag{7}$$

where $Q(s)$ is an analytical function of s given by (4).

Granted that the zero condition (7) is an implicit equation which has to be solved for $\underline{s}$ by trial-and-error, once an outlying zero has been found, as assumed, the actual values taken by the variable s and by the trascendent function $Q(s)$ are to be considered as known (as definite data) and the relationship between these values can be extracted from Equations (3), (5), and (7) by straightforward algebraic operations (of course taking into due account the constraints that any solution must comply with).

Indeed, Equation (7) is formally an algebraic equation of the second degree in s, once the actual value of $Q(s)$ (which has to be assumed as known as if it were actually evaluated in each of the four points) be substituted for the symbol $Q(s)$. Let us explicit the definition of the four H-DLVP hypothetical zero-points

$$s_1 = \frac{1}{2} + \xi + i.Y, \quad s_2 = \bar{s}_1 = \frac{1}{2} + \xi - i.Y, \quad s_3 = \frac{1}{2} - \xi + i.Y,$$

$$s_4 = \bar{s}_3 = \frac{1}{2} - \xi - i.Y, \tag{8}$$

and take one of the two c.c. pairs, e.g., s_1, s_2 (homologous developments can be carried out for the other pair), it is necessarily

$$s_1^2 - s_1 + Q(s_1) = 0, \quad s_2^2 - s_2 + Q(s_2) = 0, \tag{9}$$

where of course, since $s_2 = \bar{s}_1$, it is $Q(s_2) = Q(\bar{s}_1) = \overline{Q}(s_1)$.

Let us develop the implications of Equation (9) for one of the assumed outliers, e.g., for $s = s_1 = \frac{1}{2} + \xi + i.Y$ by posing

$$Q(s_1) = A + i.2.B \text{ with } A, B \in \mathbb{R} \text{ and } \neq 0. \tag{10}$$

From the first of the two equations in (9), it comes

$$-\frac{1}{4} + \xi^2 - Y^2 + A + i.2.(Y.\xi + B) = 0, \tag{11}$$

and equating to 0 the real and imaginary parts of the left-hand side of Equation (11) it comes

$$A = \frac{1}{4} - \xi^2 + Y^2 \text{ and } B = -\xi.Y. \tag{12}$$

Therefore, equation (9) written for the assumedly outlying zero-point 1 becomes, with obvious passages

$$s_1^2 - s_1 + Q(s_1) = 0, \tag{13}$$

and

$$\begin{aligned} s_1 &= \frac{1}{2} \pm \sqrt{\frac{1}{4} - Q(s_1)} = \frac{1}{2} \pm \sqrt{\frac{1}{4} - A - i.2.B} \\ &= \frac{1}{2} \pm \sqrt{\xi^2 - Y^2 + i.2.\xi.Y} = \frac{1}{2} \pm (\xi + i.Y), \end{aligned} \tag{14}$$

i.e., we get the assumedly outlying zero-points 1 and 4, which do not form a c.c. pair unless $\xi = 0$. Likewise, operating the same developments for the assumedly outlying zero-point 2, it comes

$$Q(s_2) = Q(\bar{s}_1) = \overline{Q}(s_1) = Q(s_3) = A - i.2.B, \tag{15}$$

which analogously to Equations (10) to (12) yields

$$s_2 = \frac{1}{2} \pm \sqrt{\frac{1}{4} - Q(s_2)} = \frac{1}{2} \pm \sqrt{\frac{1}{4} - A + i.2.B}$$

$$= \frac{1}{2} \pm \sqrt{\xi^2 - Y^2 - i.2.\xi.Y} = \frac{1}{2} \pm (\xi - i.Y), \tag{16}$$

i.e., we get the two outlying zero-points 2 and 3, which do not form a c.c. pair unless $\xi = 0$. Turning at last to the remaining two assumedly outlying zero-points 3 and 4 it comes, for point 3;

$$Q(s_3) = Q(\bar{s}_4) = \overline{Q}(s_1) = A - i.2.B, \tag{17}$$

which analogously to Equations (15) and (16) yields

$$s_3 = \frac{1}{2} \pm \sqrt{\frac{1}{4} - Q(s_3)} = \frac{1}{2} \pm \sqrt{\frac{1}{4} - A - i.2.B}$$

$$= \frac{1}{2} \pm \sqrt{\xi^2 - Y^2 + i.2.\xi.Y} = \frac{1}{2} \pm (\xi - i.Y), \tag{18}$$

identical with the Equation (16); i.e., we get once more the two outlying zero-points 2 and 3, which do not form a c.c. pair unless $\xi = 0$. For the last assumedly outlying zero-point 4 the same equation as (14) holds. Therefore, we get the same implication, i.e., $\xi = 0$.

Summing up, from Equations (10) to (18), the following conditions have to be complied with:

$$s_1 = \frac{1}{2} + (\xi + i.Y) \to \overline{s_1} = s_2 = \frac{1}{2} + (\xi - i.Y),$$

$$s_3 = \frac{1}{2} - (\xi - i.Y) \to s_4 = \frac{1}{2} - (\xi + i.Y), \tag{19}$$

$$A = \frac{1}{4} + Y^2, \quad B = 0, \quad \xi = 0, \quad Q = A = \frac{1}{4} + Y^2 \in \mathbb{R}. \tag{20}$$

4. Conclusion

The authors deem that the originality of their approach consists in keeping trace, throughout all the chain of their logical and algorithmic inferences, of the nature of the two components of the zero-condition (7). A crucial role is seen to be played by the property of invariance of the algebraic component $s.(s-1)$ of this zero-condition with respect to the exchange $s \leftrightarrow (1-s)$. Also, the structure of the Gram-Backlund extension of the Zeta function, as well as the use of Euler-McLaurin summation through integration was very important in determining the structure of the zero-condition (7).

A sort of 'proof by absurd' was then derived by working out the necessary implications of the assumption that an outlying zero was found, together with its three more outlying companions foreseen by Hadamard-De La Vallee Poussin. In this way, a contradictory statement followed, which could only be avoided by recognizing that one must necessarily accept the second of Equation (20), $\xi = 0$, which means that the assumed c.c. double pair of H-DLVP outlying zeroes must collapse onto a single pair located on the C.L.. The conclusion that no non-trivial zero of the Zeta function can fall away from the C.L. seemed at this point unescapable[3].

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[3]In extreme synthesis, it could be said that the constraint confining the c.c. pairs of zero-points to be found only on the C.L. stems from two independent requirements:

(1) The symmetry feature $Q(s) = Q(1-s) = s.(1-s)$.

(2) The inference that for the existence of a pair of c.c. outliers it should be:

$$Q(s) = Q(\bar{s}) = s.\bar{s} = Q \in \mathbb{R}.$$

Indeed, since $1 - s \neq \bar{s}$ unless $s + \bar{s} = 1 + 2.\xi = 1$, it follows necessarily that the two requirements are incompatible unless $\xi = 0$.

However, the implications of this conclusion (unless it be falsified by a deeper analysis) are so important, and the means used to reach it so startlingly simple, that the authors would hesitate to claim that theirs can be considered a fault-free proof. They were intrigued by their own line of thought, which appears as a suggestive one which probably is worth of being analyzed with mathematical methods more powerful than those within their knowledge. They remain deeply interested and fascinated by the Riemann hypothesis, and are willing and ready to collaborate with some professional mathematician who could happen to be interested in pursuing the matter further.

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