ON THE EQUIVALENCE PROBLEM FOR GEOMETRIC STRUCTURES, II

ANTONIO KUMPERA

Campinas State University
Campinas, SP
Brazil
e-mail: antoniokumpera@hotmail.com

Abstract

This paper is a continuation of Part I where the general setup was developed. Here we discuss the general equivalence problem for geometric structures and provide criteria for the equivalence, local and global, of transitive structures. Cartan's Flag Systems illustrate the theory as a major example and, finally, some attention though little is given to non-transitive structures with regular orbits, i.e., intransitivity classes.

1. Introduction

In the Part I, we recalled the theory of Differential Invariants as conceived by Sophus Lie and placed it in a context best suited for the study of the equivalence problem for geometric structures. It is interesting to add some more remarks proper to what we shall discuss in the sequel and, firstly, let us recall what the Finiteness Theorem for the differential invariants, proved in Part 1, brings to the equivalence problem. A germ of a structure S being fixed, we are interested in the

formal equivalence, general or restricted, of S with the germs of neighbouring structures or those eventually located on other manifolds. We are as well interested in global equivalences, these however offering several additional difficulties of quite a different nature since globalization is always a hard topological task. Let us now place ourselves back into the neighborhoods $\mathcal{W}_k$ as described in the statement of the Finiteness Theorem of the Part I. The constancy of the k-th order differential invariants provides us with a necessary and sufficient condition for the k-th order equivalence, namely: The k-jet of a structure S' is equivalent to the k-jet of S if and only if the k-th order differential invariants assume the same values on both jets, these two jets belonging therefore to the same orbit in $\mathcal{W}_k$. Similarly, two infinite jets of structures, elements of lim proj $\mathcal{W}_k$, are equivalent whenever they are so at every finite order. Since from an integer μ onwards the higher order differential invariants are all obtained by iterated formal admissible derivations of those of order $\leq \mu$, we infer that the infinitely many equivalence conditions are all consequences of those of order $\leq \mu$ and, moreover, just of a finite number of them, namely a finite fundamental system of such invariants. In the next section, we shall examine the formal equivalence problem for the formally transitive structures and, in later sections, more specific types of structures will also be discussed. As for the first mentioned structures, we shall see that the formal equivalence takes place if and only if the restrictions of the differential invariants of orders $\leq \mu$, to the flows of the same order associated to these structures, assume the same values and, further, the restrictions of the derived invariants to the flows of higher orders vanish. We shall see as well that when the flows of orders $\leq \mu$ associated to the germs of two structures are transverse to the trajectories - the opposite of transitivity - then a necessary and sufficient condition for the formal equivalence is the existence of a germ of local diffeomorphism φ of P such that the

restriction of every invariant of order $\leq \mu$ to one of these jets of structure be equal to the restriction to the other jet composed with φ, the same property thereafter holding for all the invariants of higher order.

It now only remains to determine criteria under which the hypotheses H_2 and H_3, stated in Part I, Section 9, are verified and it is precisely here that the specific nature of the prolongation spaces E intervenes. As for the hypothesis H_2, it will be realised, for example, upon requiring the local regularity, at a given order k, of the distributions Δ_k and $\Delta_{k,k+1}$ together with the 2-acyclicity (or even involutivity) of $\Delta_{k-1,k}$ and arguing as in Quillen's criterion (while assuming of course that 1-acyclicity already holds on a sequence of open sets $\mathcal{W}_k$, this reviving now the hypothesis H_1). A more elaborate analysis (see [5]) shows the possibility of linking the δ-complex associated to the kernels $\Delta_{k-1,k}$ with the corresponding δ-complex associated to the kernels of the spaces $\widetilde{J}_k(\mathcal{L}_\ell)$ and consequently, in view of the prolongation spaces structure, with that constructed by means of the kernels of the spaces $J_{\ell+k}\mathcal{L}$, the later introducing however a shifting, by one unit, in the cohomology groups. This matter will be discussed in more detail in later sections. However, it is also worthwhile to mention that we can employ, in this context, the method of characters by extending the reasoning of [6], Section 24, and relate the *characters* of the kernels of $J_{\ell+k}\mathcal{L}$ with those of $\Delta_{k-1,k}$.

Finally concerning the hypothesis H_3, not much can be simplified in the specific case of prolongation spaces. Apparently, the most efficacious criteria are provided, on the one hand, by the Proposition 21.5 of [6] that involves the bracket of formal and holonomic derivations and, on the other, by the Proposition 25.4 that relies on the explicit formula (25.5) which, surprisingly, can already be found in Lie's work. The hypotheses

underlying the last proposition are no other than conditions of transversality, in the strict sense, of the k-th order flows of local sections S of the fibration $E \to P$ with the trajectories of Δ_k, such conditions being entirely antipodal to a formal transitivity hypothesis. We finally remark that it is possible to transcribe all the results stated above in the case where (E, π, P, p) is no more than a prolongation space relative to an infinitesimal Lie pseudo-algebra $\mathcal{L}$ ([2]).

2. Subordinate Prolongation Spaces
and Formal Transitivity

Let us begin by stating a very naive lemma, consequence of the functoriality properties of the prolongation operation, that nevertheless is at the basis of everything that follows.

Lemma 1. *Let (E, π, P, p) be a finite prolongation space, ξ be a local vector field defined on P and φ be a local diffeomorphism such that $\alpha(\xi) = \alpha(\varphi)$. Under these conditions, $(p\varphi)_* p\xi = p(\varphi_* \xi)$.*

Let us next assume that the prolongation space has finite order ℓ. In order to render notations easier, we denote by $Z \mapsto Z_h$ the projection $J_k E \to J_h E$ and by $Z \to \overline{Z}_{k-h}$ the semi-holonomic inclusion into $J_{k-h}(J_h E)$, the same notations applying as well to $\prod_k P$. We observe that the elements of $J_{k-h}(J_h E)$ can be canonically identified with the $(k-h)$-th order holonomic contact elements of dimension n that are transverse to the fibration $J_h E \longrightarrow P$, the contact element $\overline{Z}_{k-h}$ being issued at the point Y_h. This being so, the groupoid $\prod_{\ell+k+1} P$ operates to the left (or to the right if one prefers so) on

(a) $J_m E$, $m \leq k + 1$, by means of the prolongation spaces law, in view of the Lemma 2, Section 2 in [9] or else by the left action (1) (loc.cit., pg. 5, Part I), and the projection $\prod_{\ell+k+1} P \longrightarrow \prod_{\ell+k} P$.

(b) $J_{m-h}(J_h E)$, $m \leq k+1$, by applying twice the above lemma.

(c) $J_{\ell+k} TP$, by the standard action.

(d) $J_{k-h}(TJ_h E)$, thanks to the left action Λ_{k+h} (loc. cit.) followed by

the standard action hence, in particular, on $TJ_k E$.

These actions are compatible with all the fibrations and it is quite evident that (a) and (b) are co-variant by means of the canonical inclusion. Furthermore, the preceding lemma implies that the infinitesimal prolongation operation is compatible with these actions. In fact, taking $Y \in \prod_{\ell+k+1} P$, $j_{\ell+k} \xi(y) \in J_{\ell+k} TP$, and $Y(j_{\ell+k} \xi(y)) = j_{\ell+k} \eta(y)$ then, for all $Z \in J_h E$ composable with Y, the element transformed by Y via the action (d) of the jet $j_{k-h}(p_h \xi)(Z)$ is equal to $j_{k-h}(p_h \eta)(Y(Z))$. Moreover, we can extend the action of $\prod_{\ell+k+1} P$ to $J_{\ell+k} TP \times_P J_h E$ acting separately on each factor. The previous considerations can now be summarized by the following statement (cf. (7), [9], where we replace $\overline{\lambda}_k$ by $\overline{\lambda}_{k+h}$, $\ell + k$ by $\ell + k + h$ and $TJ_k E$ by $J_h(TJ_k E)$).

Lemma 2. *The mapping*

$$\lambda_{h+(k-h)} : J_{\ell+k} TP \times_P J_h E \longrightarrow J_{k-h}(TJ_h E)$$

is a differential co-variant with respect to the action by $\prod_{\ell+k+h} P$ or, in other words, this mapping commutes with both actions.

In particular, we derive the following conclusions: Let $Y \in \prod_{\ell+k+1} P$, $Z \in J_{k+1} E$, $Y(j_{\ell+k} \xi(y)) = j_{\ell+k} \eta(y')$ and assume further that $p_h \xi$ is tangent of order $k - h$ to the contact element (of order $k - h + 1$) $\overline{Z}_{(k+1)-h} \in J_{k+1-h}(J_h E)$. Then the vector field $p_h \eta$ is tangent of order $k - h$ to the contact element $\overline{W}_{(k+1)-h}$, $W = Y(Z)$, since Y transforms $\overline{Z}_{(k+1)-h}$ onto $\overline{W}_{(k+1)-h}$ and $j_{k-h} p_h \xi(Z_h)$ onto $j_{k-h} p_h \eta(W_h)$.

Lemma 3. *The action of $\prod_{\ell+k+1} P$ is compatible with the tangency relations between prolongued vector fields and contact elements.*

Let us now introduce the (eventually singular) vector bundles, with base space $J_k E$,

$$\mathbf{R}^0_{\ell+k} = \{(Y, Z) \mid Y \in (\mathbf{R}^0_{\ell+k})_Z, \ Z \in J_k E\}, \tag{1}$$

where

$$(\mathbf{R}^0_{\ell+k})_Z = \{j_{\ell+k}\xi(y) \in J_{\ell+k}TP \mid (p_k\xi)_Z = 0, \ y = \alpha(Z)\},$$

and

$$\mathfrak{g}_{\ell+k} = \ker (\mathbf{R}^0_{\ell+k} \to \mathbf{R}^0_{\ell+k-1}), \tag{2}$$

the choice of the notations being justified by the fact that the linear isotropies as well as the symbols of orders $\ell + k$ of a given structure S of species E can simply be obtained as the inverse images of these bundles via the flow $j_k S$ or, in other terms, by the relations

$$\mathbf{R}^0_{\ell+k}(S) = (j_k S)^{-1}\mathbf{R}^0_{\ell+k}, \quad \mathfrak{g}_{\ell+k}(S) = (j_k S)^{-1}\mathfrak{g}_{\ell+k}. \tag{3}$$

The following sequence, where the third arrow is the morphism (7) in [9], is of course exact:

$$0 \longrightarrow \mathbf{R}^0_{\ell+k} \longrightarrow J_{\ell+k}TP \times_P J_k E \xrightarrow{\overline{\lambda_k}} \Delta_k \longrightarrow 0. \tag{4}$$

The preceding lemmas show in particular that each term of this sequence is invariant by the corresponding action of $\prod_{\ell+k+1} P$. Moreover, the action of $\prod_{\ell+k+1} P$ induced on $\mathbf{R}^0_{\ell+k}$ factors to $\prod_{\ell+k} P$ since it is no other than the parts of the orders $\leq \ell + k$ that operate on the total isotropy $J^0_{\ell+k}TP$. Finally, we observe that the action of $\prod_{\ell+k} P$ on $J^0_{\ell+k}TP \times_P J_k E$ leaves invariant the sub-space

$$\mathfrak{g}_{\ell+k} \subset (S^{\ell+k}T^*P \otimes TP) \times_P J_k E$$

and the induced action on the tensorial term factors, via $\prod_{\ell+k} P \longrightarrow \prod_1 P$, since it is only the first order part that operates on the symbols.

Lemma 4. *For all $Z \in J_k E$, the symbol $(\mathfrak{g}_{\ell+k})_Z$ is the k-th algebraic prolongation of $(\mathfrak{g}_\ell)_{\beta(Z)}$ and, consequently, $\mathfrak{g}_{\ell+k}$ is entirely determined by $\mathfrak{g}_\ell$ via the relation $\mathfrak{g}_{\ell+k} = \mathfrak{g}_\ell^{(k)} \otimes_E J_k E$, where $\mathfrak{g}_\ell^{(k)}$ is the k-th algebraic prolongation of $\mathfrak{g}_\ell$.*

In fact, it suffices to choose a section S of E such that $Z = j_k S(y)$ and apply the Proposition 4 in [9]. The previous discussion also entails the following lemma:

Lemma 5. *Let Ω_0 be an orbit of $\prod_\ell P$ in E. For every k, the restriction $\mathfrak{g}_\ell \mid \beta^{-1}(\Omega_0)$ is a vector bundle of constant rank on the manifold $\beta^{-1}(\Omega_0)$. Further, taking any two points $Z, Z' \in \beta^{-1}(\Omega_0)$ and assuming that the jet $Y \in \prod_\ell P$ transforms $\beta(Z)$ in $\beta(Z')$, then $(\mathfrak{g}_{\ell+k})_Z$ is transformed onto $(\mathfrak{g}_{\ell+k})_{Z'}$ by the tensorial extension of the 1-jet Y_1, projection of Y, considered as a linear transformation $T_{\alpha(Z)}P \longrightarrow T_{\alpha(Z')}P$.*

Corollary 1. *For all k, the Spencer δ-cohomology complexes associated to the symbols $(\mathfrak{g}_{\ell+k})_Z$ and $(\mathfrak{g}_{\ell+k})_{Z'}$ taken at any two arbitrary points of $\beta^{-1}(\Omega_0)$ are isomorphic by means of the restriction of an isomorphism, of the total complexes, produced by the linear part of an element belonging to $\prod_\ell P$ that sends $T_{\alpha(Z)}P$ into $T_{\alpha(Z')}P$. Consequently, the symbols have the same homological properties along the sub-manifold $\beta^{-1}(\Omega_0)$. In particular, when S is a section of E taking its values in $\beta^{-1}(\Omega_0)$, then the symbol $\mathfrak{g}_{\ell+k}(S)$ is a vector bundle of constant rank and each fibre has the same homological properties.*

Lemma 6. *Let Ω_0 be an orbit of $\prod_\ell P$ in E. Then, for every k, the vector bundle $\mathfrak{g}_\ell^{(k)} \mid \Omega_0$ is of constant rank and is invariant by the action of the groupoid $\prod_\ell P$ on $(S^{\ell+k}T^*P \otimes TP) \times_P \Omega_0$, the action on the first term factoring by $\prod_\ell P \longrightarrow \prod_1 P$. Furthermore, $\mathfrak{g}_{\ell+k} \mid \beta^{-1}(\Omega_0) = \mathfrak{g}_\ell^{(k)}$*

$\times_{\Omega_0} \beta^{-1}(\Omega_0)$. *Moreover, if S is a section of E taking its values in Ω_0, then the groupoid $\mathcal{R}(S)$ is transitive on $\alpha(S)$ and the linear action of $\rho_1(\mathcal{R}(S))$ leaves invariant the vector bundle $\mathfrak{g}_{\ell+k}(S)$.*

Lemma 7. *Let $\mathbf{F}_k$ be an orbit, in $J_k E$, of the finite or infinitesimal action (i.e., an orbit Ω_k or ω_k) of $\prod_{\ell+k} P$ and let us write $\mathbf{F}_h = \rho\mathbf{F}_k$, $0 \le h \le k$. Under these conditions:*

(i) $\mathbf{F}_h$ is a finite or infinitesimal orbit of $J_h E$ according to the nature of the corresponding k-orbit.

(ii) The action of $\prod_{\ell+k+1} P$ on $J_{\ell+h} TP \times_P \mathbf{F}_h$, $0 \le h \le k$, leaves invariant the sub-bundles $\mathbf{R}^0_{\ell+h} \mid \mathbf{F}_h$ and $\mathfrak{g}_{\ell+k} \mid \mathbf{F}_h$, the induced action then factoring to $\prod_{\ell+k} P$, and operating transitively in the base space $\mathbf{F}_h$. The isotropies $(\mathbf{R}^0_{\ell+h})_Z$ and $(\mathbf{R}^0_{\ell+h})_{Z'}$ as well as the symbols $(\mathfrak{g}_{\ell+h})_Z$ and $(\mathfrak{g}_{\ell+h})_{Z'}$, at any two points $Z, Z' \in \mathbf{F}_h$, are therefore isomorphic by means of the action by the elements of $\prod_{\ell+k} P$, resp., $\prod_1 P$.

(iii) The action of $\prod_{\ell+k+1} P$ on $T J_h E \mid \mathbf{F}_h$, $0 \le h \le k$, leaves invariant the sub-bundle $\Delta_h \mid \mathbf{F}_h$, the fibres at any two arbitrary points being isomorphic via the above action and, moreover, $\Delta_h \mid \mathbf{F}_h = T\mathbf{F}_h$.

(iv) The restrictions $\mathbf{R}^0_{\ell+h} \mid \mathbf{F}_h$, $\mathfrak{g}_{\ell+k} \mid \mathbf{F}_h$ and $\Delta_h \mid \mathbf{F}_h = T\mathbf{F}_h$ are vector bundles of constant rank.

By all that has already been said, the proof is obvious. Let us mention however that, in view of the exactness of the sequence (4), the vector bundles $\mathbf{R}^0_{\ell+h}$ and Δ_h are *simultaneously* regular i.e., when one of them is regular then the other is also necessarily regular. We next state (without proof!) a rather long lemma that outlines all the basic techniques to be used from here onwards.

Lemma 8. *The hypotheses being those of the preceding lemma, for each solution S of $\mathbf{F}_k$ namely, a section S of the prolongation space E such that $j_k S$ takes its values in $\mathbf{F}_k$, the equations $\mathcal{R}_{\ell+h}(S)$ and $\mathbf{R}_{\ell+h}(S)$, $0 \leq h \leq k$, are transitive in $\alpha(S)$ and the bundles $\mathbf{R}_{\ell+h}(S)$, $\mathbf{R}_{\ell+h}^0(S)$ and $\mathfrak{g}_{\ell+h}(S)$ are all of constant rank. Further, $\mathcal{R}_{\ell+h}(S)$ is a closed and locally trivial Lie (differentiable) sub-groupoid, its differentiable structure being regularly embedded in $\prod_{\ell+h} \alpha(S)$. The non-linear isotropy $\mathcal{R}_{\ell+h}^0(S)$ is a closed and locally trivial sub-bundle of $\prod_{\ell+h}^0 \alpha(S)$, each fibre being a closed Lie sub-group of the total group. The projection $\rho_{\ell+h-1}\mathcal{R}_{\ell+h}(S)$ is a locally trivial Lie sub-groupoid of $\prod_{\ell+h-1} \alpha(S)$ and the non-linear symbol*

$$\mathfrak{g}_{\ell+h}(S) = \ker(\mathcal{R}_{\ell+h}(S) \longrightarrow \rho_{\ell+h-1}\mathcal{R}_{\ell+h}(S))$$

is a locally trivial affine sub-bundle (unless when $\ell = 1$ and $h = 0$, in which case the fibre is a linear group) of the total symbol above $\rho_{\ell+h-1}\mathcal{R}_{\ell+h}(S)$. Finally, the groupoid $\mathcal{R}_{\ell+h}(S)$ leaves invariant the sub-bundles $\mathbf{R}_{\ell+h}^0(S)$ and $\mathfrak{g}_{\ell+h}(S)$ via the standard actions on $J_{\ell+h}^0 TP$ and $S^{\ell+h}T^*P \otimes TP$, respectively. Inasmuch, $\mathcal{R}_{\ell+h}(S)$ leaves invariant the isotropy $\mathcal{R}_{\ell+h}^0(S)$ via the action on $\prod_{\ell+h}^0 P$ defined by conjugation, as well as the non-linear symbol $\mathfrak{g}_{\ell+h}(S)$ via the standard action on the total symbol defined by the translations to the left or to the right. We deduce that any two fibres of $\mathcal{R}_{\ell+h}^0(S)$ and $\mathfrak{g}_{\ell+h}(S) \mid Id(= \alpha(S))$ are always isomorphic as Lie groups and that those of $\mathfrak{g}_{\ell+h}(S)$ are also isomorphic as homogeneous spaces.

Lemma 9. *The data as well as the hypotheses being those of the preceding lemma, we also have the following equalities, for $0 \leq h \leq k - 1$, $\mathbf{R}_{\ell+h+1}^0(S) = p\mathbf{R}_{\ell+h}^0(S)$ and $\mathcal{R}_{\ell+h+1}(S) = p\mathcal{R}_{\ell+h}(S)$, where p is the standard prolongation operator for partial differential equations.*

We finally introduce the (eventually singular) vector bundle with basis $J_{k+1}E$

$$\mathbf{R}_{\ell+k} = \{(Y, Z) \mid Y \in (\mathbf{R}_{\ell+k})_Z, \, Z \in J_{k+1}E\}, \tag{5}$$

where

$$(\mathbf{R}_{\ell+k})_Z = \{j_{\ell+k}\xi(z) \in J_{\ell+k}TP \mid (\rho_k\xi)_Z = \overline{Z}_{(k+1)-1}\}.$$

Observe that $\overline{Z}_{(k+1)-1} \in J_1(J_kE)$ is a first order contact element namely, a transverse vector sub-space of $T_{Z_k}J_kE$. The choice of these notations is justified by the relation

$$\mathbf{R}_{\ell+k}(S) = (j_{k+1}S)^{-1}\mathbf{R}_{\ell+k}. \tag{6}$$

Lemma 10. *The fibre bundle $\mathbf{R}_{\ell+k}$ is invariant under the action of $\prod_{\ell+k+1} P$ on $J_{\ell+k}TP \times_P J_{k+1}E$. If $\mathbf{F}_{k+1}$ is a finite or infinitesimal orbit in $J_{k+1}E$, then the restriction $\mathbf{R}_{\ell+k} \mid \mathbf{F}_{k+1}$ is of constant rank and any two fibres are isomorphic by the action of some element in $\prod_{\ell+k+1} P$.*

Let us recall (cf., the Theorem 2 in [9]) that each trajectory Ω_k is a locally trivial sub-bundle of J_kE isomorphic to the quotient space $(\prod_{\ell+k} P)_y / H(Y)$. Inasmuch, an infinitesimal trajectory ω_k is a locally trivial sub-bundle since it is isomorphic to the quotient of the connected component of the unit y in $(\prod_{\ell+k} P)_y$ by the isotropy (we shall be careful to assume P connected otherwise ω_k is just a bundle whose basis is the connected component of P containing the point y). We can thus consider any trajectory $\mathbf{F}_k$, finite or infinitesimal, as a non-linear partial differential equation (a non-linear differential system) of order k on the fibration (and prolongation space) $\pi : E \longrightarrow P$ that we shall call *a fundamental equation of order k.*

Lemma 11. *Let $\mathbf{F}_{k+1}$ be a fundamental equation, $\mathbf{F}_k$ its projection at k-th order and assume that $\mathbf{F}_{k+1} \cap p\mathbf{F}_k \neq \phi$. Under these conditions, $\mathbf{F}_{k+1} \subset p\mathbf{F}_k$ and therefore $p\mathbf{F}_k$ is a union of trajectories.*

Lemma 12. *A fundamental equation* $\mathbf{F}_k$ *is 1-integrable namely, the property* $\rho_k(p\mathbf{F}_k) = \mathbf{F}_k$ *holds, if and only if* $p\mathbf{F}_k \neq \phi$ *(for every fibre). This condition being verified, the prolongation* $p\mathbf{F}_k$ *is a locally trivial affine sub-bundle of* $J_{k+1}E \longrightarrow \mathbf{F}_k$, *where E is assumed to be a finite prolongation space.*

Let us denote by $p_h\mathbf{F}_k$ the prolongation of order h of $\mathbf{F}_k$. Arguing as previously, we can also show the

Corollary 2. *A fundamental equation* $\mathbf{F}_k$ *verifies the extended equality* $\rho_k(p_h\mathbf{F}_k) = \mathbf{F}_k$ *if and only if* $p_h\mathbf{F}_k \neq \phi$ *and, whenever this condition is verified, the following properties also hold*:

(i) *For every* $\mu \leq h$, $p_\mu\mathbf{F}_k$ *is a locally trivial (and non-void) sub-bundle of* $J_{k+\mu}E \longrightarrow \mathbf{F}_k$ *saturated by the orbits of order* $k + \mu$, *finite or infinitesimal according to the nature of* $\mathbf{F}_k$.

(ii) $p_\mu(p_\eta\mathbf{F}_k) = p_{\mu+\eta}\mathbf{F}_k$, $\mu + \eta \leq h$.

We observe, however, that the property $p_h\mathbf{F}_k \neq \phi$ does not imply, in general, the h-integrability of $\mathbf{F}_k$, namely, the property

$$\rho_\mu(p_{\mu+1}\mathbf{F}_k) = p_\mu\mathbf{F}_k, \mu \leq h.$$

The above property only implies the 1-integrability.

Let us next examine the tangent symbol of a fundamental equation. On the one hand, it is clear that this symbol at the point $Z \in J_kE$ is simply the contact element

$$(\Delta_{k-1,k})_Z = \ker\left[(\Delta_k)_Z \longrightarrow (\Delta_{k-1})_{\rho_{k-1}Z}\right],$$

$$(\Delta_{k-1,k})_Z \subset \left[(S^kT^*P \otimes VE) \times_E J_kE\right]_Z = \left[S^kT^*_{\alpha Z}P \otimes V_{\beta Z}E\right] \times \{Z\},$$

and, on the other, we see immediately that the diagram below is commutative and exact (cf., the second diagram at the outset of Section 3

in [9]). Curiously enough, this diagram cannot be completed everywhere by surjectivities, as is so common in commutative diagrams, since structures do not behave always as nicely as one would like to and the problem resides in the lack of *h-integrability* or, stated equivalently, in the difficulty of approximating what we really look for namely, *the integrability*.

Lemma 13. *The first vertical sequence in (7) is surjective at the end if and only if the first horizontal sequence verifies the same property*.

$$
\begin{array}{ccccc}
(7) \quad 0 & & 0 & & 0 \\
\downarrow & & \downarrow & & \downarrow \\
0 \longrightarrow \mathfrak{g}_{\ell+k} \longrightarrow & (S^{\ell+k}T^*P \otimes TP) \times_P J_kE & \xrightarrow{\ell_k} & \Delta_{k-1,k} \\
& & & \downarrow \\
\downarrow & & \downarrow & & |\downarrow \\
& & & \downarrow \\
0 \longrightarrow \mathbf{R}^0_{\ell+k} \longrightarrow & J_{\ell+k}TP \times_P J_kE & \xrightarrow{\lambda_k} & |\Delta_k & \longrightarrow & 0 \\
& & & \downarrow \\
\downarrow \rho_{\ell+k-1} & & \downarrow & & |\downarrow \\
0 \to \mathbf{R}^0_{\ell+k-1} \times_P J_kE \longrightarrow & J_{\ell+k-1}TP \times_P J_kE & \xrightarrow{\lambda_{k-1}} & \Delta_{k-1} \times_{J_{k-1}E} J_kE & \longrightarrow & 0 \\
& & \downarrow & & \downarrow \\
& & 0 & & 0
\end{array}
$$

In fact, the above *snake arrow* tells us that the following sequence is exact:

$$
0 \to (S^{\ell+k}T^*P \otimes TP) \times_P J_kE \xrightarrow{\ell_k} \Delta_{k-1,k} \to \mathrm{coker}\ \rho_{\ell+k-1} \to 0.
$$

The Lemma 1 can now be extended to the following assertion:

Lemma 14. *Let* (E, π, P, p) *be a finite prolongation space of order* ℓ, ζ *a local vector field defined on* E, φ *a local diffeomorphism of P and we assume that* $\alpha(p\varphi) = \alpha(\zeta)$. *Then,* $[p_k(p\varphi)]_* \, p_k\zeta = p_k[(p\varphi)_*\zeta]$.

We infer the following corollaries by taking also into account the Theorem 13.1 as well as the Proposition 14.2 in [6].

Lemma 15. *The groupoid* $\prod_{\ell+k+1} P$ *operates differentiably on the spaces* $\widetilde{J}_k TE$, $TJ_k E$ *and* $(S^k T^* P \otimes VE) \times_E J_k E$. *The Lie sub-fibration* $\widetilde{J}_k VE$ *as well as the vector sub-bundle* $VJ_k E$ *are invariant by this action and the following exact sequences*:

$$0 \longrightarrow J_k^0 TP \times_P J_k E \xrightarrow{\Sigma_k} \widetilde{J}_k TE \xrightarrow{p_k} TJ_k \longrightarrow 0,$$

and

$$0 \longrightarrow (S^k T^* P \otimes VE) \times_E J_k E \xrightarrow{\epsilon_k} TJ_k E \xrightarrow{T\rho_{k-1,k}} TJ_{k-1} E \times_{J_{k-1}E} J_k E \longrightarrow 0$$

are co-variant. Furthermore, the action on the term $S^k T^* P \otimes VE$ *factors via* $\rho_{\ell+1} : \prod_{\ell+k+1} P \longrightarrow \prod_{\ell+1} P$ *and that on the term* $J_k^0 TP$ *via* $\rho_k : \prod_{\ell+k+1} P \longrightarrow \prod_k P$.

Lemma 16. *Let* $\mathbf{F}_k$ *be a fundamental equation and* $\mathbf{F}_{k-1}$ *its projection. By means of the canonical identification,*

$$\Delta_{k-1,k}|\mathbf{F}_k = \ker [T\mathbf{F}_k \longrightarrow T\mathbf{F}_{k-1}]$$

is a locally trivial vector sub-bundle of $(S^k T^* P \otimes VE) \times_E \mathbf{F}_k$ *invariant under the action of* $\prod_{\ell+k} P$. *When* $k \geq 1$ *and when* $Z, Z' \in \mathbf{F}_k$ *and* $Y \in \prod_{\ell+k} P$ *are such that* $Y(Z) = Z'$, *then Y transforms* $(\Delta_{k-1,k})_Z$ *in* $(\Delta_{k-1,k})_{Z'}$ *by means of the tensorial extension of the* $(\ell+1)$*-jet* $Y_{\ell+1}$, *projection of Y and consider as a linear map* $T_{\beta(Z)}E \longrightarrow T_{\beta(Z')}E$. *In the case where* $k = 0$,

$$\Delta_{-1,0}|\mathbf{F}_0 = \ker\left[T\pi : T\mathbf{F}_0 \longrightarrow TP\right] \subset VE|\mathbf{F}_0,\, Y \in \textstyle\prod_\ell P,$$

and we can take any jet $Y_{\ell+1} \in \prod_{\ell+1} P$ projecting upon Y. When $\mathbf{F}_k$ is an infinitesimal trajectory, we shall take care to restrict the action just to the open subset of $\prod_{\ell+k} P$ that leaves invariant $\mathbf{F}_k$.

We next consider a pair $\mathbf{F}_{k+1}$ and $\mathbf{F}_k$ of fundamental equations, $k \geq 0$, verifying $\rho_k(\mathbf{F}_{k+1}) = \mathbf{F}_k$. We shall study the geometrical properties of the fibration $\mathbf{F}_{k+1} \longrightarrow \mathbf{F}_k$ inasmuch as a sub-fibration of the affine bundle $J_{k+1}E \longrightarrow \mathbf{F}_k$ above the base space $\mathbf{F}_k$ (more precisely, it would be convenient to replace the term $J_{k+1}E$ by $\rho_k^{-1}(\mathbf{F}_k) \subset J_{k+1}E$, where $\rho_k : J_{k+1}E \longrightarrow J_kE$). Since $\mathbf{F}_{k+1}$ is an orbit, this fibration is obviously a locally trivial sub-bundle, any two fibres being isomorphic by the action of the elements belonging to $\prod_{\ell+k+1} P$. We next recall ([6], Section 19) that $J_{k+1}E \longrightarrow J_kE$ is an affine bundle, the underlying vector bundle being equal to $(S^{k+1}T^*P \otimes VE) \times_E J_kE$ and the affine action defined as follows: The point $Z \in J_kE$ being fixed, a vector $v \in S^{k+1}T_y^*P \otimes V_zE$, $y = \alpha(Z)$, $z = \beta(Z)$, defines by means of the canonical identification ([6], Proposition 14.2) a vector field $\nu(v)$ along the fiber A_Z of $J_{k+1}E$ above the point Z. This vector field admits a global 1-parameter group (φ_t) of affine transformations and the affine action of v on $W \in A_Z$ is defined by $W + v = \varphi_1(W)$. We thus see that the set $\{\nu(v) \mid v \in S^{k+1}T_y^*P \otimes V_zE\}$ is the abelian Lie algebra of all the infinitesimal translations of the affine space A_Z. If we examine again the diagram (14.1) in [6], we shall observe that the pair (v, W) identifies to an element in $\widetilde{J}_{k+1}VE$ that projects onto $(0, Z)$ by the morphism $\widetilde{\rho}_{k,k+1}$ (see the first line of the diagram (14.1)). This means however, in setting $W = j_{k+1}\sigma(y)$ and $Z = \rho_k(W)$, that (v, W) identifies to a jet

$j_{k+1}(\zeta \circ \sigma)(y)$, where ζ is a vertical vector field on E null to order k (i.e., it vanishes up to order k) along the image of the section σ namely, $j_k(\zeta \circ \sigma)(y) = 0$. Equivalently, ζ is a vertical vector field, on E, *tangent* to order k along the image of σ and at the point $\sigma(y)$. This being so, we see readily, and according to the diagram (14.1), that $\nu(v)_W = (p_{k+1}\zeta)_W$.

Let us now return to the fibration $\rho_k : \mathbf{F}_{k+1} \longrightarrow \mathbf{F}_k$. Since $\mathbf{F}_{k+1}$ is an orbit under the action of $\prod_{\ell+k+1} P$ on $J_{k+1}E$, the fibre $(\mathbf{F}_{k+1})_W$, $W \in \mathbf{F}_k$, is the orbit of an arbitrary point $W' \in (\mathbf{F}_{k+1})_W$ under the action of the sub-group $K(W)$ of $(\prod_{\ell+k+1}^0 P)_y$, inverse image of the isotropy $H(Z)$ by the projection $\rho_{k, k+1}$. Consequently, the fibre $(\mathbf{F}_{k+1})_W$ is isomorphic to the homogeneous space $K(W)/H(Z)$ that, in general, is not connected. However, we can easily determine its connected components since they are simply the orbits of the connected component of the unity in $K(W)$ or, equivalently, the orbits of the infinitesimal action of the Lie algebra $k(W)$ of the group $K(W)$. We shall therefore study the algebra $q(W)$ of the vector fields on A_Z, images by the infinitesimal action of $k(W)$. We first remark that $T_Z(\mathbf{F}_{k+1})_W = (\Delta_{k, k+1})_Z$ and that consequently, every vector $w \in T_Z(\mathbf{F}_{k+1})_W$ is obtained as follows: We take a local vector field ξ of P defined in a neighbourhood of y and consider its prolongation $p_{k+1}\xi = p_{k+1}(p\xi)$. Under these conditions, $(p_{k+1}\xi)_W \in (\Delta_{k, k+1})_W$ if and only if $T\rho_k(p_{k+1}\xi)_Z = (p_k\xi)_Z = 0$, $Z = \rho_k(W)$, which means precisely that $j_{\ell+k+1}\xi(y) \in k(W)$. Moreover, the condition $(p_k\xi)_Z = 0$ also means, in writing $Z = j_k\sigma(y)$, $z = \sigma(y)$, that (cf., [6], p.317, the remark just following the corollary):

(a) $p\xi(z) = 0$ and

(b) $p\xi$ is tangent to order k along $im \; \sigma$ at the point z.

Since we have nullity at the point z, we infer that the k-th order tangency condition along the section σ and at the point z only depends on the k-th order jet of σ at the point y or, in other terms, $p\xi$ will also be tangent up to order k along another section τ and at the point z as soon as $j_k\tau(y) = j_k\sigma(y)$. The vector field ξ determines, by $(k+1)$-st order prolongation, a vector field $p_{k+1}\xi$ tangent to A_Z and we shall write

$$\widetilde{\nu}(\xi) = p_{k+1}\xi|A_Z = p_{k+1}(p\xi)|A_Z.$$

Since the prolongation operation preserves the brackets, i.e.,

$$[p_{k+1}(p\xi),\ p_{k+1}\zeta] = p_{k+1}([p\xi,\ \zeta]),$$

where ζ is an arbitrary local vector field on E, then

(1) $[p_{k+1}\xi,\ p_{k+1}\zeta]|A_Z = [\widetilde{\nu}(\xi),\ \nu(v)] = p_{k+1}([p\xi,\ \zeta])|A_Z$,

(2) $[p\xi,\ \zeta]$ is a vertical vector field on E,

(3) $[p\xi,\ \zeta]$ is null up to order k along the section σ at the point $\sigma(y)$ since ζ verifies the same property and $p\xi$ is tangent to order k along σ and vanishes at the point $\sigma(y)$.

We then infer that $[\widetilde{\nu}(\xi),\ \nu(v)] = \nu(v')$, where $v' \in S^{k+1}T_y^*P \otimes V_z E$ is the element that, for all $W \in A_Z$, determines the identification

$$(v',\ W) \equiv (j_{k+1}([p\xi,\ \zeta] \circ \tau)(z) \in J_{k+1}VE,\ W = j_{k+1}\tau(z)).$$

The above relation shows that any vector field $\widetilde{\nu}(\xi)$ belongs to the normalizer, in $W(A_Z)$, of the algebra of infinitesimal transformations of the affine space A_Z. Consequently, $q(W)$ is a finite dimensional Lie algebra, sub-algebra of the affine infinitesimal transformations of A_Z. Since the above argumentation only puts in evidence the infinitesimal transformations, we see readily that the preceding results remain valid when $\mathbf{F}_{k+1}$ and $\mathbf{F}_k$ are replaced just by infinitesimal orbits.

Theorem 1. *Let $(\mathbf{F}_{k+1}, \mathbf{F}_k)$ be a pair of fundamental equations verifying $\rho_k \mathbf{F}_{k+1} = \mathbf{F}_k$. The connected components of the fibre $(\mathbf{F}_{k+1})_Z$, $Z \in \mathbf{F}_k$, are the orbits of the finite action, on A_Z, of the connected affine Lie group whose Lie algebra is equal to $q(Z)$. If $Y \in \prod_{\ell+k+1} P$ and if $\rho_k Y(Z) = W$, then $Y[(\mathbf{F}_{k+1})_Z] = (\mathbf{F}_{k+1})_W$ and the affine transformation $(J_{k+1}E)_Z \longrightarrow (J_{k+1}E)_W$, induced by Y, transforms $q(Z)$ in $q(W)$ conjugating the corresponding affine groups.*

Let us next assume that the first line of the diagram (7) is surjective at the end at a point $Y \in (\mathbf{F}_{k+1})_Z$. Since $\mathbf{F}_{k+1}$ is an orbit, the Lemma 1 implies that the same property will also hold at any other point $Y' \in (\mathbf{F}_{k+1})_Z$. Furthermore, the surjectivity of $(\ell_{k+1})_Y$ shows that any vector in $T_Y(\mathbf{F}_{k+1})_Z = (\Delta_{k,k+1})_Y$ can be obtained by the prolongation of a vector field ξ satisfying the property $j_{\ell+k+1}\xi(y) \in S^{\ell+k+1}T_y^*P \otimes T_yP$. On account of the two commutative diagrams in the beginning of the Section 3 in Part I ([9]), we finally infer that $p_{k+1}\xi|A_Z$ is *an infinitesimal translation.*

Corollary 3. *When ℓ_{k+1} is surjective at a point $W \in \mathbf{F}_{k+1}$ then it is also surjective at any other point of $\mathbf{F}_{k+1}$ and the Lie algebra $q(Z)$, $Z \in \mathbf{F}_k$, contains a sub-algebra $t(Z)$ of infinitesimal translations whose orbit issued from a point of $(\mathbf{F}_{k+1})_Z$ is equal to that of $q(Z)$. Each connected component of $(\mathbf{F}_{k+1})_Z$ is an affine sub-space of A_Z.*

Corollary 4. *When ℓ_{k+1} is surjective at a point in $\mathbf{F}_{k+1}$ and if further the fibre $(\mathbf{F}_{k+1})_Z$ is connected, then $\rho_k : \mathbf{F}_{k+1} \longrightarrow \mathbf{F}_k$ is a locally trivial affine sub-bundle of $J_{k+1}E \longrightarrow \mathbf{F}_k$, any two fibres being linearly isomorphic under the action of $\prod_{\ell+k+1} P$.*

Lemma 17. *Let $(\mathbf{F}_k)_{k \geq \mu}$ be a family of fundamental equations verifying the following properties: $\rho_k \mathbf{F}_{k+1} = \mathbf{F}_k$ and $\mathbf{F}_{k+1} \cap p\mathbf{F}_k \neq \phi$. Under these conditions, $\mathbf{F}_{k+1} \subset p\mathbf{F}_k$ for all k and there exists an integer*

μ_0 *such that* $\mathbf{F}_{k+1}$ *is an open sub-bundle of* $p\mathbf{F}_k$, $k \geq \mu_0$. *The integer* μ_0
is the order of stability (1-*acyclicity*) *of the Spencer* δ-*complex associated
to the tangent symbols of the equations* $\mathbf{F}_k$ (*and their prolongations*).

The proof is always essentially the same and so will be omitted.

We now recall that a fundamental equation does not have necessarily any solution at all. Fortunately or not, there are *non-analytic* though *involutive* differential systems that do not possess any solution! As for the fundamental equations, if ever they possess a germ of a solution passing by one of their points then the same will hold for all the other points and the equation will be *completely* (*or everywhere*) *integrable*. It is worthwhile to state the following corollary, consequence of the preceding lemma, though containing some repetitions.

Corollary 5. *If* $(\mathbf{F}_k)_{k \geq \mu}$ *is a family of integrable fundamental equations and if further the following property holds*: $\rho_k \mathbf{F}_{k+1} = \mathbf{F}_k$, *then* $\mathbf{F}_{k+1} \subset p\mathbf{F}_k$ *for all* k *and* $\mathbf{F}_{k+1}$ *is an open sub-bundle of* $p\mathbf{F}_k$ *for* $k \geq \mu_0$. *We can even assume that* $\mu_0 = 0$ *since the projection of an integrable fundamental equation is also integrable.*

Let us now return to the study of symbols. We know that $\mathfrak{g}_{\ell+k+1}$ is always the algebraic prolongation of $\mathfrak{g}_{\ell+k}$ but this is not the case, in general, for the kernels $\Delta_{k-1,k}$. Nevertheless,

Lemma 18. *Let* (Z_k) *be an element of* $J_\infty E$ *and let us assume, for* $k \geq \mu$, *that the morphism* ℓ_k *in the first horizontal sequence of* (7) *is surjective at the point* Z_k. *Under these conditions*:

(1) $\Delta_{k,k+1} \subset p\Delta_{k-1,k}$ *for* $k \geq \mu$, *and*

(2) *the Spencer* δ-*complex corresponding to the family* $(\Delta_{k-1,k})_{k \geq \mu}$ *is* p-*acyclic at order* μ_p (*i.e., for* $k \geq \mu_p$) *if and only if the complex constructed with the aid of the family* $(\mathfrak{g}_{\ell+k})_{k \geq 0}$ *is* $(p+1)$-*acyclic at order* $\ell + \mu_p - 1$.

As for the proof, it will suffice to recall ([9], Section 3) that the family (ℓ_k) is a natural transformation of the corresponding δ-cohomology complexes compatible with the algebraic prolongation operations performed on the principal parts and proceed with the *diagram chasing* below by confronting the term $\wedge^{q+1}T^*P \otimes \mathfrak{g}_{\ell+k-1}$ with $\wedge^q T^*P \otimes \Delta_{k-1,k}$. As for the notations, we suggest to have a glance at [12], p.83.

$$0 \to \wedge^{q-1}T^*P \otimes \mathfrak{g}_{\ell+k+1} \to \wedge^{q-1}T^*P \otimes (S^{\ell+k+1}T^*P \otimes TP) \xrightarrow{Id \otimes \ell_{k+1}} \wedge^{q-1}T^*P \otimes \Delta_{k,k+1} \to 0$$

$$\downarrow \delta \qquad\qquad \downarrow \delta \qquad\qquad \downarrow \delta$$

$$0 \to \wedge^q T^*P \otimes \mathfrak{g}_{\ell+k} \longrightarrow \wedge^q T^*P \otimes (S^{\ell+k}T^*P \otimes TP) \xrightarrow{Id \otimes \ell_k} \wedge^q T^*P \otimes \Delta_{k-1,k} \longrightarrow 0$$

$$\downarrow \delta \qquad\qquad \downarrow \delta \qquad\qquad \downarrow \delta$$

$$0 \to \wedge^{q+1}T^*P \otimes \mathfrak{g}_{\ell+k-1} \to \wedge^{q+1}T^*P \otimes (S^{\ell+k-1}T^*P \otimes TP) \xrightarrow{Id \otimes \ell_{k-1}} \wedge^{q+1}T^*P \otimes \Delta_{k-2,k-1} \to 0$$

$$\downarrow \delta \qquad\qquad \downarrow \delta$$

$$0 \to \wedge^{q+2}T^*P \otimes \mathfrak{g}_{\ell+k-2} \to \wedge^{q+2}T^*P \otimes (S^{\ell+k-2}T^*P \otimes TP)$$

$$(8)$$

An entirely analogous argument, where we replace $\Delta_{k-1,k}$ by h_k, shows as well the following lemma that transcribes what we can expect when the surjectivity only occurs at the level μ.

Lemma 19. *Let $Z \in J_\mu E$ be an element such that the first horizontal sequence in (7) is surjective. Setting $h_\mu = (\Delta_{\mu-1,\mu})$ and indicating by h_k, $k \geq \mu$, the $(k-\mu)$-th algebraic prolongation of h_μ, then h_k is p-acyclic if and only if $\mathfrak{g}_{\ell+k-1}$ is $(p+1)$-acyclic.*

Theorem 2. *The hypotheses being those of the Lemma 14, let us further assume that there exists a formal solution $(Z_k)_{k \geq \mu}$ of the fundamental system $\mathbf{F} = (\mathbf{F}_k)_{k \geq \mu}$ (i.e., $Z_k \in \mathbf{F}_k$ and $\rho_k Z_{k+1} = Z_k$) such that the first sequence of (7) is surjective at the end along (Z_k). This being so, we infer that*

(1) *the same property of surjectivity also holds for any other formal solution* $W = (W_k)$ *of* F,

(2) $\mathbf{F}_{k+1} = p\mathbf{F}_k$ *for* $k \geq \mu_0$, *and*

(3) $\mu_0 = \sup \{\mu, \mu_2\}$, *where* μ_2 *is the integer such that the symbol* $\mathfrak{g}_{\ell+k}$, $k \geq \mu_2 - 1$, *becomes 2-acyclic.*

We recall that the symbol $\mathfrak{g}_{\ell+k}(Z_k)$, $Z_k \in \mathbf{F}_k$, is entirely determined by $\mathfrak{g}_\ell(Z_0)$ and that the homological properties of this symbol are uniform along a trajectory $\mathbf{F}_k$ in E (cf., the Proposition 4 in [9], Section 6 and the Corollary 1).

Remark. Under the hypotheses of the preceding theorem, we have assumed the surjectivity of the morphisms ℓ_k, for $k \geq \mu$, in view of being able to guarantee the stability of the symbols tangent to the fundamental equations $\mathbf{F}_k$ by means of the 2-*acyclicity* of the algebraic symbols $\mathfrak{g}_{\ell+k}$. If we let down this property, very useful in applications, it will suffice to assume the surjectivity of the morphisms ℓ_k for $k \geq \mu_0 + 1$.

Corollary 6. *The hypotheses being those of the last theorem, the solutions of* $\mathbf{F}_{\mu_0}$ *coincide with the simultaneous solutions of the* (*infinitely many*) *fundamental equations* $\mathbf{F} = (\mathbf{F}_k)_{k \geq \mu}$ *i.e., with the local sections S of E satisfying the conditions im $j_k S \subset \mathbf{F}_k$, $k \geq \mu$.*

Corollary 7. *Let* $\mathbf{F} = (\mathbf{F}_k)_{k \geq \mu}$ *be a family of integrable fundamental equations verifying the property* $\rho_k \mathbf{F}_{k+1} = \mathbf{F}_k$ *and let us assume further that* ℓ_k, $k \geq \mu_0 + 1$, *is surjective along a formal solution of* $\mathbf{F}$, μ_0 *being the integer that stabilizes the symbols tangent to the equations* $\mathbf{F}_k$. *Under these conditions, we claim that* $\mathbf{F}_{k+1} = p\mathbf{F}_k$ *for all* $k \geq \mu_0$ *and that the solutions of* $\mathbf{F}_{\mu_0}$ *coincide with the simultaneous solutions of the system* F. *We also conclude thereafter that any particular solution of a given equation* $\mathbf{F}_k$ *is as well a simultaneous solution.*

The sorites being terminated (21 pages), let us return to geometry and try to do some adequate work. We thus begin by considering a finite prolongation space (E, π, P, p) of order ℓ and take a local or global section S of E namely, a structure of species E. We shall say that S is *homogeneous* or *transitive in the base space* if the germs of S at any two points $y, y' \in \alpha(S)$ (the source of S) are always equivalent. In other terms, there exists a germ of local diffeomorphism ϕ of P with source y and target y' such that

$$\phi(\underline{S}_y) \stackrel{\text{def}}{=} p\phi \circ \underline{S}_y \circ \phi^{-1} = \underline{S}_{y'},$$

where $\underline{S}_y$ denotes the germ of S at the point y. Inasmuch, we shall say that S is *homogeneous of order k* when the k-jets of S at any two arbitrary points $y, y' \in \alpha(S)$ are k-th order equivalent which means that there exists an invertible jet $Y \in \prod_{\ell+k} P$ such that $Y(j_k S(y)) = j_k S(y')$, $y = \alpha(Y)$, $y' = \beta(Y)$, and where, by definition, $Y(j_k S(y)) = Y^{(k)} \cdot j_k S(y) \cdot Y^{-1}$, $Y^{(k)}$ being the invertible k-jets on E that corresponds, by prolongation, to the $(\ell + k)$-jet Y on P. We shall say, finally, that S is *formally homogeneous* when the above k-th order condition is verified for all the integers k.

We now assume that S is homogeneous of order k which means that the set $\{j_k S(y) | y \in \alpha(S)\}$ is contained in a single trajectory, denoted by $\Omega_k(S)$, under the action of the pseudo-group $\Gamma_{\ell+k}$ on $J_k E$, natural prolongation, k-th order, of the *general* pseudo-group $\Gamma(P)$ of all local diffeomorphisms of the manifold P. Furthermore, when $\alpha(S)$ is connected, the above set $\Omega_k(S)$ is also contained in a single trajectory $\omega_k(S)$ of the infinitesimal action $\mathcal{L}_{\ell+k}$ since these latter trajectories are the connected components of the finite trajectories of $\Gamma_{\ell+k}$. Each $\Omega_k(S)$ is a finite fundamental equation of order k and each $\omega_k(S)$ is an infinitesimal fundamental equation of the same order. When S is

formally homogeneous, we shall denote by $\Omega(S) = (\Omega_k(S))$ the family of all the respective fundamental equations. If, further, $j_k S$ takes all its values in a given infinitesimal trajectory, we denote by $\omega(S) = (\omega_k(S))$ the corresponding family.

Proposition 1. *Let S be a homogeneous structure of order k and $\Omega_k(S)$ its finite fundamental equation of the same order. Then $\Omega_{h+1}(S) \subset p\Omega_h(S)$ for every integer $h < k$. If, furthermore, S is formally homogeneous, then there exists an integer μ_0 such that $\Omega_{h+1}(S)$ is an open sub-bundle of $p\Omega_h(S)$ for any $h \geq \mu_0$. The integer μ_0 is the order of stability (1-acyclicity) of the Spencer δ-complex associated to the symbols tangent to the equation $\Omega_k(S)$. If, moreover, the symbol (i.e., the fibre) of $\Omega_{h+1}(S)$ above $\Omega_h(S)$ is, along $j_\infty S(y)$ and for $h \geq \mu$, an affine sub-space of the total symbol (more generally, if each connected component of the symbol is an affine sub-space), then $\Omega_{h+1}(S) = p\Omega_h(S)$ for $h \geq h_0 = \sup \{\mu_0, \mu\}$. This being so, the solutions of $\Omega_{h_0}(S)$ coincide with the simultaneous solutions of $\Omega(S)$. The same properties also remain valid for $\omega_{h+1}(S)$ and $\omega(S)$, the integers μ_0 and h_0 being the same as those for $\Omega(S)$.*

It is clear that the solutions S' of $\Omega_k(S)$ are precisely the structures of species E that are k-th order equivalent to the *model S*, the k-th order equivalence taking place for all the couples of points $y \in \alpha(S)$ and $y' \in \alpha(S')$. The structures S' are of course homogeneous of order k and any structure of this type whose jet at a single point is k-th order equivalent to a k-jet of S is a structure k-th order equivalent to S. Likewise, the simultaneous solutions of the family $\Omega(S)$ are the structures formally equivalent to the model S, such structures being all formally homogeneous. We shall say that a structure is *connected* whenever $\alpha(S)$ is connected, a *connected component* of an arbitrary structure S being, by definition, the restriction of S to a connected component of $\alpha(S)$. Its image is a connected component of *im S*.

Proposition 2. *Let S be a formally homogeneous structure. Then, the local or global structures of species E and formally equivalent to the model S are the solutions of* $\Omega(S)$*. Moreover, when the hypotheses of the last proposition concerning the affine nature of the symbols of* $\Omega - h(S)$ *are verified, the formal equivalence is then a consequence of the equivalence at order* h_0*. Finally, when* $\omega(S)$ *is defined* (*which occurs especially when S is connected*), *the connected structures formally equivalent to the model S are the solutions of* $\omega(S)$*.*

The system $\Omega(S)$ is, according to the terminology of [13], *the fundamental differential system* for the structures of species E that are formally equivalent to S. In his article, the author considers the model structure defined on a manifold P_0 eventually distinct from P. In our case, we simply identify P_0 to an open set of P since it is always possible to *transfer* the model given on P_0 onto a model defined on an open set of P without, for that matter, modifying the equivalence relation.

We could hereafter contemplate in defining the *reduced* fundamental system and in transcribing some of the results of [13]. However, we leave such transcriptions, not all together evident, to the care of the reader since, at present, we are inclined to examine other important aspects of the theory.

Let us first observe that the different notions of homogeneity introduced above refer essentially to the pseudo-group $\Gamma(S)$ of all the automorphisms of S as well as to the groupoids $\mathcal{R}_{\ell+k}(S)$. In fact, S is *homogeneous* when $\Gamma(S)$ is transitive, *homogeneous of order* k when the groupoid $\mathcal{R}_{\ell+k}(S)$ is transitive and finally *formally homogeneous* when the transitivity of $\mathcal{R}_{\ell+k}(S)$ occurs for all k. We can therefore introduce the corresponding infinitesimal notions, namely, we can say that S is *infinitesimally homogeneous* when $\mathcal{L}(S)$ is transitive, k-th *order infinitesimally homogeneous* when $\mathbf{R}_{\ell+k}(S)$ is transitive, i.e.,

$\beta(\mathbf{R}_{\ell+k}(S))_y = T_yP$, for all $y \in P$, and *formally infinitesimally homogeneous* when the last condition is verified for all k. When S is only defined on an open set $\mathcal{U}$, we shall simply replace, in the above definitions, the manifold P by the open set $\mathcal{U}$.

Let S be a structure of species E and let us inquire for conditions rendering the target map $\beta : (\mathbf{R}_{\ell+k}S)_y \longrightarrow T_yP$ surjective. For this, we set $Y = j_{\ell+k}S(y)$ and recall that

$$(\mathbf{R}_{\ell+k}S)_y = \{j_{\ell+k}\xi(y) \in J_{\ell+k}TP \mid (p_k\xi)_Y \in T_Y(im\, j_kS)\}.$$

However, since $\xi_y = \beta(j_{\ell+k}\xi(y)) = \alpha_*(p_k\xi)_Y$, $\alpha : J_kE \longrightarrow P$, we see promptly that the surjectivity condition is given by $T_Y(im\, j_kS) \subset (\Delta_k)_Y$. Consequently, a structure S is infinitesimally homogeneous of order k if and only if $im\, j_kS$ is an integral sub-manifold of the distribution (differential system) Δ_k. Taking into account the integrability (involutivity) of Δ_k ([9], Theorem 2) and the fact that the integral leaves of Δ_k are the connected components of the trajectories Ω_k of $\Gamma_{\ell+k}$, we obtain the following result:

Proposition 3. *The following statements are equivalent*:

(1) *S is infinitesimally homogeneous of order k.*

(2) *Each connected component of $im\, j_kS$ is contained in an infinitesimal trajectory of $\mathcal{L}_{\ell+k}$.*

(3) *Each connected component of $im\, j_kS$ is contained in a finite trajectory of $\Gamma_{\ell+k}$.*

(4) *Each connected component of $im\, S$ is homogeneous of order k.*

Corollary 8. *The following statements hold*:

(1) *Every connected structure that is infinitesimally homogeneous of order k is also homogeneous of order k and every homogeneous structure of order k is also infinitesimally homogeneous of the same order.*

(2) *Let S be a connected and infinitesimally homogeneous model structure of order k. Then the class of all the connected structures S' equivalent to order k to the model S is given by the connected solutions of the fundamental equation $\omega_k(S)$. Each such equivalent structure S' is infinitesimally homogeneous of order k.*

(3) *Let S be a connected and infinitesimally formally homogeneous model structure. Then the class of all the connected structures S' formally equivalent to the model S is given by the connected solutions of the infinitesimal fundamental system $\omega(S)$ or, when the hypotheses of the Proposition 1 are verified, by the connected solutions of $\omega_{h_0}(S)$ in which case the formal equivalence is a consequence of the equivalence at order h_0. Any solution S' is infinitesimally formally homogeneous.*

We next discuss the notion of *transitivity*. The structure S is said to be *transitive of order k* or k-th order *transitive* when it is homogeneous of order k and, further, when the projections $\mathcal{R}_{\ell+k}(S) \longrightarrow \mathcal{R}_{\ell+h}(S)$, $0 \le h \le k$, are all surjective (in fact, surmersions on account of the Lemma 8). The structure S is said *formally transitive* when the above conditions are verified for all k and, finally, just *transitive* when it is formally transitive and, moreover, when $\mathcal{R}_{\ell+k}(S) = J_{\ell+k}\Gamma(S)$. In other terms, this means that any equivalence of finite order among the jets of S is actually realised (achieved) by the jet of a local automorphism of S. In essentially the same way, we can introduce the notions of k-th order infinitesimal transitivity, formal infinitesimal transitivity and, for short, transitivity by simply replacing $\Gamma(S)$ by $\mathcal{L}(S)$ and $\mathcal{R}_{\ell+k}(S)$ by $\mathbf{R}_{\ell+k}(S)$.

We observe that the notions of k-th order and formal finite or infinitesimal transitivity are invariant by k-th order and formal equivalences, respectively. Thus, if we start with a formally transitive resp., k-th order model, every solution of the fundamental system $\Omega(S)$ resp., $\Omega_k(S)$ is a formally transitive resp., k-th order structure. The same

remarks apply of course to the infinitesimal context taking, however, the care in choosing for S a formally homogeneous and infinitesimally formally transitive resp., k-th order infinitesimally transitive structure.

Let us now make abstraction of the model. It is clear that a structure S is homogeneous of order k if and only if this structure is a solution of a fundamental equation Ω_k. Moreover, each connected component of S is a solution of an infinitesimal equation ω_k contained in Ω_k. Inasmuch, S is infinitesimally homogeneous of order k if and only if each connected component is a solution of a finite or infinitesimal fundamental equation of order k, though these equations may vary along with the connected component. We have analogous conclusions in the formal context, the number of equations being infinite unless the hypotheses of the Proposition 1 be verified.

We next inquire on the equations bearing on the finite jets of a structure S displaying their k-th order or their formal transitivity attributes and start examining the infinitesimal aspects that will curiously place in evidence a new element namely, the *Medolaghi-Vessiot equations* (called improperly *Vessiot-Medolaghi* by several authors).

Let us be given a structure S of species E, let us fix an integer k and let us pose ourselves the following problem. What are the conditions to which are bound the finite jets $j_{k+1}S(y)$, $y \in \alpha(S) \subset P$ so that

(a) the projection $\mathbf{R}_{\ell+k+1}(S) \longrightarrow \mathbf{R}_{\ell+k}(S)$ be surjective (1-integrability at order $\ell + k$) and

(b) $\mathbf{R}_{\ell+k}(S) \longrightarrow TP$ be surjective (infinitesimal homogeneity of order k).

We observe immediately that the second condition can be replace, in view of (a), by the surjectivity of $\mathbf{R}_{\ell+k+1}(S) \longrightarrow TP$. Moreover, we can replace the above problem by the following equivalent problem:

(a_1) $\mathbf{R}^0_{\ell+k+1}(S) \longrightarrow \mathbf{R}^0_{\ell+k}(S)$ be surjective and

(b_1) $\mathbf{R}_{\ell+k+1}(S) \longrightarrow TP$ be surjective,

since the *small* diagram hereafter is commutative and exact.

$$0 \longrightarrow \mathbf{R}^0_{\ell+k+1}(S) \longrightarrow \mathbf{R}_{\ell+k+1}(S) \longrightarrow TP \longrightarrow 0$$
$$\downarrow \qquad\qquad \downarrow \qquad\qquad \|$$
$$0 \longrightarrow \quad \mathbf{R}^0_{\ell+k}(S) \quad \longrightarrow \quad \mathbf{R}_{\ell+k}(S) \longrightarrow TP \longrightarrow 0$$
$$\vdots \qquad\qquad \vdots$$
$$0 \qquad\qquad 0$$

We first examine the problem (a_1). The relations (3) lead us to introduce the equation Θ_{k+1} of order $k+1$ and defined on the fibration $E \longrightarrow P$,

$$\Theta_{k+1} = \{Z \in J_{k+1}E \mid (\mathbf{R}^0_{\ell+k+1})_Z \longrightarrow (\mathbf{R}^0_{\ell+k})_{\rho(Z)} \longrightarrow 0 \text{ is exact}\}.$$

The following lemma is evident on account of the Lemma 1.

Lemma 20. *The equation Θ_{k+1} is invariant under the finite action of $\Gamma_{\ell+k+1}$ as well as under the infinitesimal action of $\mathcal{L}_{\ell+k+1}$ or, in other terms, Θ_{k+1} is a union of trajectories. Inasmuch, this equation is invariant under the left action of the groupoid $\prod_{\ell+k+1} P$ on $J_k E$ and the infinitesimal action of the algebroid $J_{\ell+k+1}TE$. The solutions of Θ_{k+1} are the local or global structures of species E verifying the condition a_1.*

Since the condition b_1 is no other than the $(k+1)$-st order infinitesimal homogeneity condition, we derive the following result:

Proposition 4. *A structure of species E verifies the conditions (a) and (b) if and only if each of its connected components is a solution of a finite or infinitesimal fundamental equation contained in the equation Θ_{k+1}. The connected structures satisfying (a) and (b) are the connected solutions of the infinitesimal fundamental equations contained in Θ_{k+1}.*

We thus see that there exists a whole family of differential equations of order $k+1$ on the fibration $E \longrightarrow P$ solving the proposed problem. Whereas the fundamental equations are well behaved with respect to

prolongations, this does not occur with the equations Θ_{k+1}. In fact, there does not exist, *a priori*, any plausible relation between Θ_{k+1} and $p\Theta_k$. Inasmuch, we can even add that there does not exist any apparent relation between $\rho_k(\Theta_{k+1})$ and Θ_k, all depending on the specific structure of the prolongation space E. We shall therefore search for the desired results with the help of the theorem of Quillen.

With this in mind, let us first recall that this theorem strongly relies on the 2-acyclicity of the symbols and the Corollary 1 tells us that the homological properties of the symbols $\mathfrak{g}_{\ell+k}$ are uniform along the inverse image of an orbit of degree zero hence, in particular, along a fundamental equation $\mathbf{F}_k$. On the other hand, if S is a solution of $\mathbf{F}_k$, S takes its values in the orbit of order zero $\mathbf{F}_0 = \beta(\mathbf{F}_k)$. Consequently, when $\mathbf{F}_k$ is integrable then $\mathbf{F}_k \subset J_k\mathbf{F}_0$ and this leads us naturally to only consider the prolongation space $(\mathbf{F}_0, \pi, P, p)$ of order ℓ that will be called *subordinate* to (E, π, P, p) and whose prolongation structure is simply obtained by restricting the structure of E (cf., [9], Theorem 2). When $\mathbf{F}_0$ is a finite trajectory, the prolongation laws of E admit natural restrictions. Quite to the contrary, when the trajectory is infinitesimal, the infinitesimal prolongation law restricts without any problem while the finite law will have to be restricted, at each order k, not only to the open sub-groupoid of $\prod_{\ell+k} P$, composed by those finite jets that preserve the k-th order transverse contact elements of dimension n tangent to $\mathbf{F}_0$ but also to the solutions of this sub-groupoid. In particular, the α-connected component $(\prod_{\ell+k} P)_0$ is contained in this sub-groupoid. It is clear that $J_k\mathbf{F}_0$ is invariant in J_kE (always respecting the above restrictions concerning the infinitesimal orbits) and we shall denote by $\Theta_{k+1}(\mathbf{F}_0)$ and $\mathfrak{g}_{\ell+k}(\mathbf{F}_0)$ the equation Θ_{k+1} and the symbol $\mathfrak{g}_{\ell+k}$, respectively, when restricted to $J_{k+1}\mathbf{F}_0$ and $J_k\mathbf{F}_0$. The restricted symbol has the same homological properties everywhere. Lastly, we shall call $\mathbf{F}_0$-*admissible* any fundamental equation contained in Θ_{k+1} and thus find the so claimed Medolaghi-Vessiot equations ([15], p.436, Equations (58) and (59)).

Theorem 3. *Let* (E, π, P, p) *be a finite prolongation space of order* ℓ, $(\mathbf{F}_0, \pi, P, p)$ *the prolongation space subordinate to an orbit* $\mathbf{F}_0$ *and* $\eta_2 = \eta_2(\mathbf{F}_0)$ *the integer from whereon the symbol* $\mathfrak{g}_{\ell+k}(\mathbf{F}_0)$, $k \geq \eta_2$, *becomes 2–acyclic. The linear equation* $\mathbf{R}_{\ell+k}(S)$ *associated to any solution of an* $\mathbf{F}_0$*-admissible fundamental equation of order* $k + 1 \geq \eta_2 + 1$ *is then transitive and formally integrable i.e., formally transitive. Moreover, when* $\eta_\infty = \eta_\infty(\mathbf{F}_0)$ *is the integer where from the restricted symbol* $\mathfrak{g}_{\ell+k}(\mathbf{F}_0)$, $k \geq \eta_\infty$, *becomes involutive, then the equation* $\mathbf{R}_{\ell+k}(S)$, $k \geq \eta_\infty$, *is involutive.*

The argument is as follows. S being a solution of a fundamental equation of order $k + 1$, we know (cf., the Lemma 8) that $\mathbf{R}_{\ell+k}(S)$ as well as $\mathbf{R}_{\ell+k+1}(S)$ are regular equations and that $\mathbf{R}_{\ell+k+1}(S) = p\mathbf{R}_{\ell+k}(S)$. Furthermore, the symbol $\mathfrak{g}_{\ell+k}(S) = (j_k S)^{-1}\mathfrak{g}_{\ell+k}(\mathbf{F}_0)$ is 2-acyclic and the morphism $\mathbf{R}_{\ell+k+1}(S) \longrightarrow \mathbf{R}_{\ell+k}(S)$ is surjective since S is a solution of the Medolaghi-Vessiot equations of order $k + 1$. We thus re-encounter the hypotheses of Quillen's theorem ([12]) and consequently $\mathbf{R}_{\ell+k}(S)$ is formally integrable. We also observe that $\mathbf{R}_{\ell+k+2}(S) = p \; \mathbf{R}_{\ell+k+1}(S)$ since the equation of order $\ell + k + 1$ is regular and the 2-acyclicity implies that $\mathbf{R}_{\ell+k+2}(S)$ is also regular. An inductive argument will then show that all the equations $\mathbf{R}_{\ell+k+h}(S)$ are regular and that $\mathbf{R}_{\ell+k+h+1}(S) = p\mathbf{R}_{\ell+k+h}(S)$, which by the way is already stated in the theorem of Quillen. We finally observe that this theorem does not require the regularity of the first two equations since it suffices in fact that $\mathbf{R}_{\ell+k}(S)$, hence also its prolongation, be defined by a differential operator, which is always the case for the equations $\mathbf{R}_{\ell+k}(S)$. The regularity is just a Black Friday extra bonus.

Corollary 9. *When* $\mathfrak{g}_\ell(\mathbf{F}_0)$ *is 2-acyclic then every solution S of an* $\mathbf{F}_0$-*admissible fundamental equation of order* 1 *is an infinitesimally formally transitive structure of species E (more so, of species* $\mathbf{F}_0$*). If, further,* $\mathfrak{g}_\ell(\mathbf{F}_0)$ *is involutive, the same will be true for the structure S, i.e., for the equation* $\mathbf{R}_\ell(S)$.

We can now consider the section $j_k S$ as being a structure of species $J_k E$, *prolongation of order* k of the structure S and the last theorem can be paraphrased by the

Corollary 10. *Let* $(\mathbf{F}_0,\ \pi,\ P,\ p)$ *be the prolongation space subordinate to an orbit* $\mathbf{F}_0$ *of* (E,π,P,p). *The* k-*th order prolongation of any* $\mathbf{F}_0$-*admissible fundamental equation of order* $k+1 \geq \eta_2 +1$ *is a structure of species* $J_k E$ (*viz. of species* J_k) *that moreover is infinitesimally formally transitive. Inasmuch, the prolongation of order* $\geq \eta_\infty$ *is an involutive structure.*

We need not play, for the prolongued structures, the Ehresmannian semi or non-holonomic game, i.e., by considering spaces like $J_h(J_{\ell+k}E)$, since we can see promptly that $\mathbf{R}_{(\ell+k)+h}(j_k S) = \mathbf{R}_{\ell+k+h}(S)$ as soon as we define these equations by the *contact order* of the prolongued vector fields with the sections defining respectively the structure.

Corollary 11. *Let* $\mathbf{F}_{k+1}$ *be an integrable fundamental equation contained in* Θ_{k+1} *and assume that* $\mathfrak{g}_{\ell+k}(\mathbf{F}_0)$, $\mathbf{F}_0 = \rho_0 \mathbf{F}_{k+1}$, *is 2-acyclic. Then, for any* $h \geq 0$, *the h-th prolongation* $p_h \mathbf{F}_{k+1}$ *is an* $\mathbf{F}_0$-*admissible fundamental equation of order* $k+h+1$, *namely, the unique integrable fundamental equation projecting onto* $\mathbf{F}_{k+1}$.

In order to prove this statement, we take an arbitrary connected solution S of $\mathbf{F}_{k+1}$, set $Y_{k+h} = j_{k+h}S(y)$, $h \geq 1$, and denote by $\mathbf{F}_{k+h}$ the fundamental equation of order $k+h$ (finite of infinitesimal according to the nature of $\mathbf{F}_{k+1}$) that contains the point Y_{k+h}. The Theorem 3 tells us that $\mathbf{R}_{\ell+k}(S)$ is transitive and formally integrable whereupon we conclude that:

(a) The structure S is formally homogeneous hence $im\ j_{k+h}S \subset \mathbf{F}_{k+h}$, i.e., $\mathbf{F}_{k+h}$ is integrable and consequently

$$\mathbf{F}_{k+h+1} \subset p\mathbf{F}_{k+h} \subset p_h\mathbf{F}_{k+1}.$$

(b) The morphism

$$(\mathbf{R}^0_{\ell+k+h})_{Y_{k+h}} \longrightarrow (\mathbf{R}^0_{\ell+k+h-1})_{Y_{k+h-1}}$$

is surjective hence $\mathbf{F}_{k+h}$ is $\mathbf{F}_0$-admissible and, on account of the Lemma 12, the mapping $(\ell_{k+h})_{Y_{k+h}}$ is also surjective. The 2-acyclicity of $\mathfrak{g}_{\ell+k}(\mathbf{F}_0)$ implies (Lemma 15) that $\mu_0 = k+1$ is equal to the integer that stabilizes the symbols tangent to the equations $\mathbf{F}_k$ and consequently (Corollary 10), $\mathbf{F}_{k+h+1} = p\mathbf{F}_{k+h}$ or, in other terms, $\mathbf{F}_{k+h+1} = p_h\mathbf{F}_{k+1}$. The uniqueness of the equation $\mathbf{F}_{k+h+1}$ is obvious since any other integrable fundamental equation of order $k+h+1$ that projects onto $\mathbf{F}_{k+1}$ is necessarily contained hence equal to $p_h\mathbf{F}_{k+1}$.

This corollary shows that given an $\mathbf{F}_0$-admissible and integrable fundamental equation $\mathbf{F}_{k+1}$, $k+1 \geq \eta_2 + 1$ (cf., the Theorem 3), there will be no place in pushing any further the calculations that is to say, search for solutions S of $\mathbf{F}_0$-admissible fundamental equations $\mathbf{F}_{k+1+h}$ projecting upon $\mathbf{F}_{k+1}$ since we shall find no other than the solutions of $\mathbf{F}_{k+1}$, no additional restriction being therefore possible.

We thus see that the local or global structures of species E admitting an *infinitesimally formally transitive* prolongation can be searched for among the solutions of the $\mathbf{F}_0$-admissible and integrable fundamental equations of orders $k+1 \geq \eta_2(\mathbf{F}_0)+1$. Moreover, this method exhausts the connected structures. In fact, if j_kS is an infinitesimally formally transitive structure, the equation $\mathbf{R}_{\ell+k}(S)$ is formally integrable and transitive. Taking, if necessary, an integer $k' > k$, this equation acquires a 2-acyclic symbol. Since, by definition, this equation verifies the

conditions (a) and (b), the section S is the solution of a fundamental equation $\mathbf{F}_{k+1}$ contained in Θ_{k+1} (Proposition 3) and thereafter is $\mathbf{F}_0$-admissible, where $\mathbf{F}_0 = \rho_0 \mathbf{F}_{k+1}$. Since $\mathfrak{g}_{\ell+k}(\mathbf{F}_0)$ is 2-acyclic, we can find the structure S by the method of the Theorem 3. When the structure is not connected, each of its components can be determined by the above method.

The preceding results involve integrable fundamental equations hence it is high time for us to examine the formal integrability of the less restrictive $\mathbf{F}_0$-admissible fundamental equations. The next lemma as well as the theorem show that, subject to reasonable hypotheses, the Medolaghi-Vessiot equations have good properties with respect to prolongations and formal integrability.

Lemma 21. *Let* $\mathbf{F}_{k+1}$ *be an* $\mathbf{F}_0$-*admissible fundamental equation verifying the two properties*: $p\mathbf{F}_{k+1} \neq \phi$ *and* $\mathfrak{g}_{\ell+k}(\mathbf{F}_0)$ *is 2-acyclic. Then* $p\mathbf{F}_{k+1}$ *is also an* $\mathbf{F}_0$-*admissible fundamental equation.*

In order to prove this rather remarkable result, we first observe that the condition $p\mathbf{F}_{k+1} \neq \phi$ entails (Corollary 2) that $p\mathbf{F}_{k+1}$ is a regular equation of order $k + 2$, namely, a locally trivial affine sub-bundle of $J_{k+2}E \longrightarrow \mathbf{F}_{k+1}$ and, in particular, that its tangent symbol $(\gamma_{k+2})_Z$, at a point $Z \in p\mathbf{F}_{k+1}$ is the algebraic prolongation of $(\Delta_{k,k+1})_Y$, $Y = \rho_{k+1}Z$, the latter being the symbol tangent to $\mathbf{F}_{k+1}$ at the point Y. We also know (Lemma 11) that $p\mathbf{F}_{k+1}$ is a union of trajectories, i.e., the trajectory $\mathbf{F}_{k+2}$ that contains the point Z is entirely contained in $p\mathbf{F}_{k+1}$. We infer that the tangent symbol $(\Delta_{k+1,k+2})_Z$ of $\mathbf{F}_{k+2}$ at the point Z is contained in $(\gamma_{k+2})_Z$. Let us now re-examine the diagram (8) at its stage $(1, k + 1)$ where we shall replace the kernel $\Delta_{k-1,k}$ by *im* ℓ_k. The fact that $\mathbf{F}_{k+1}$ be $\mathbf{F}_0$-admissible implies (Lemma 12) that *im* $(\ell_{k+1})_Y = (\Delta_{k,k+1})_Y$ and the proper definition of ℓ_k (cf., $(7)_k$) shows that *im* $(\ell_{k+2})_Z \subset (\Delta_{k+1,k+2})_Z$.

Last but not least, the 2-acyclicity of $\mathfrak{g}_{\ell+k}(\mathbf{F}_0)$ implies that the third vertical sequence in (8), i.e., the sequence

$$im\,(\ell_{k+2})_Z \xrightarrow{\delta} T_y^*P \otimes (\Delta_{k,k+1})_Y \xrightarrow{\delta} \wedge^2 T_y^* \otimes im\,(\ell_k)_Y$$

$Y = \rho_k Z$, is exact. But this means precisely that $im\,(\ell_{k+2})_Z$ is the algebraic prolongation of $(\Delta_{k,k+1})_Y$ or, in other terms, that $im\,(\ell_{k+2})_Z$ $= (\gamma_{k+2})_Z \supset (\Delta_{k+1,k+2})_Z$ and consequently that $im\,(\ell_{k+2})_Z = (\Delta_{k+1,k+2})_Z$. The trajectory $\mathbf{F}_{k+2}$ is $\mathbf{F}_0$-admissible (Lemma 12) and $\mathbf{F}_{k+2} = p\mathbf{F}_{k+1}$ since $(\ell_{k+2})_Z$ is surjective (Corollary 6) and the symbols tangent to both of these equations coincide (cf., the proof of the Lemma 14).

Theorem 4. *Let $\mathbf{F}_{k+1}$ be an $\mathbf{F}_0$-admissible fundamental equation and let us further assume that the symbol $\mathfrak{g}_{\ell+k}(\mathbf{F}_0)$ is 3-acyclic. Then $\mathbf{F}_{k+1}$ is formally integrable if and only if $p\mathbf{F}_{k+1} \neq \phi$ (on each fibre!) and, this being the case, each $p_h\mathbf{F}_{k+1}$ is an $\mathbf{F}_0$-admissible fundamental equation. If, moreover, $\mathfrak{g}_{\ell+k}(\mathbf{F}_0)$ is involutive then $\mathbf{F}_{k+1}$ is also involutive.*

The proof is almost obvious. We know, according to the Corollary 2, that $p\mathbf{F}_{k+1} \neq \phi$ if and only if $\mathbf{F}_{k+1}$ is 1-integrable and, whenever this condition is fulfilled, that $p\mathbf{F}_{k+1}$ is also a regular equation. More precisely, $p\mathbf{F}_{k+1}$ is a locally trivial affine sub-bundle of $J_{k+2}E \longrightarrow \mathbf{F}_{k+1}$. The 3-acyclicity hypothesis on $\mathfrak{g}_{\ell+k}(\mathbf{F}_0)$ entails, having in mind the Lemma 16, that the tangent symbol $\Delta_{k,k+1}$ of $\mathbf{F}_{k+1}$ is everywhere 2-acyclic. We can then complete the proof using a non-linear version of the theorem of Quillen and showing the formal integrability of $\mathbf{F}_{k+1}$ by induction on the integer h so as to propagate the result of the preceding lemma to all the prolongations (see, for a hint, [12] and [7]).

Scholium 1. When (E, π, P, p) is an analytic prolongation space and when $\mathbf{F}_{k+1}$ is an $\mathbf{F}_0$-admissible 1-integrable fundamental equation whose symbol $\mathfrak{g}_{\ell+k}(\mathbf{F}_0)$ is 3-acyclic, then $\mathbf{F}_{k+1}$ is integrable, each of its

elements being the $(k+1)$-jet of an analytic solution S. Furthermore, the solutions of any such equation are analytic structures whose prolongations of order $k+1$ are infinitesimally transitive.

Prolonguing if necessary the equation $\mathbf{F}_{k+1}$, we find an equation with involutive symbol hence the result is a consequence of *the Cartan-Kähler or the Cauchy-Kovalevskaya* (*or even perhaps the Monge-Ampère*?) theorems. The last assertion follows from the above mentioned theorems and an eventual prolongation procedure of the formally integrable equation $\mathbf{R}_{\ell+k}(S)$ so as to attain involutivity.

The condition $p\mathbf{F}_{k+1} \neq \phi$ is a point wise imposition since, for this to be so, it is necessary and sufficient that, at a point $Y \in \mathbf{F}_{k+1}$, the tangent space $T_Y\mathbf{F}_{k+1}$ contain a contact element of order 1 transverse to the fibration $J_{k+1}E \longrightarrow P$ and *holonomic*. This condition corresponds, when $k = 0$, to what the author of [14] calls *The generalized Jacobi relations*, that seem rather out of place.

Let us next show that the restrictions $\Theta_{k+1}(\mathbf{F}_0)$ of Θ_{k+1} to the subordinate prolongation spaces are closed sub-sets of $J_{k+1}\mathbf{F}_0$. We shall prove simultaneously that the restrictions $\Theta_{k+1} \cap \beta_{k+1}^{-1}(\mathbf{F}_0)$ are closed sub-spaces in $\beta_{k+1}^{-1}(\mathbf{F}_0)$, where $\beta_{k+1} : J_{k+1}E \longrightarrow J_kE$. The argument will be based on a local coordinates calculation which, in its essence, can already be found in [1], Vol. 2, Chapter V, Section 9 (*Cenni sulle ricerche di Engel-Medolaghi-Vessiot*).

Let (x^i) be a system of local coordinates defined on an open set U of P, (x^i, y^λ) an adapted system on the open set $\mathcal{U}$ of E and $(x^i, y^\lambda, y^\lambda_\alpha)_{|\alpha|\leq k}$ the corresponding system on $\beta_k^{-1}(\mathcal{U}) \subset J_kE$. Let us further denote by (z^i) the same coordinates (x^i) on U and then start our *coordinate juggling game*.

At each point $z = (z^i)$ belonging to U, the vector field

$$\xi^{\beta}_{i,z} = (1/\beta!)(x-z)^{\beta}\partial/\partial x^i, \quad (x-z)^{\beta} = (x^1 - z^1)^{\beta_1}\ldots(x^n - z^n)^{\beta_n},$$

$$1 \le |\beta| \le k,$$

vanishes at the point z and, consequently, $\epsilon^{\beta}_i(z) = j_k\xi^{\beta}_{i,z}(z) \in (J^0_k TP)_z$. We further remark that the set $\{\epsilon^{\beta}_i(z)\}$ is a basis of this Lie algebra, *the Engel basis* according to the terminology of [15] since, as soon as $\xi = \sum \xi^i \partial/\partial x^i$, then $j_k\xi(z) = \sum(\partial^{|\beta|}\xi^i/\partial x^{\beta})(z)\epsilon^{\beta}_i(z)$.

The sections $\epsilon^{\beta}_i : U \to J^0_k TP$ are differentiable and constitute a local basis of the fibre bundle $J^0_k TP|U$. Let us now place ourselves in the context of an ℓ-th prolongation space E and let us re-take the above notations with k replaced by $\ell + k$. To each section ϵ^{β}_i, $1 \le |\beta| \le \ell + k$, of $J^0_k TP$, corresponds a prolongued vector field $p_k\epsilon^{\beta}_i$ of $J_k E$ defined on the open set $\alpha_k^{-1}(U)$ and taking its values in the vector bundle $VJ_k E$. The set of all these prolongued vector fields is not, in general, linearly independent at all the points but, nevertheless, it generates the vertical distribution $\Delta_k \cap VJ_k E$. However, for any $Z \in \alpha_k^{-1}(U)$, the isotropy $(\mathbf{R}^0_{\ell+k})_Z$ is the set of all the linear combinations of the $(\ell + k)$-order Engel basis, at the point z, whose coefficients are precisely those that establish a relation of linear dependence among the prolongued vectors $p_Z\epsilon^{\beta}_i$. In order to better describe the vector space of these linear relations, let us return, with the obvious modifications, to the exact sequence (7) of [9] namely,

$$0 \to \mathbf{R}^0_{\ell+k} \to J^0_{\ell+k}TP \times_P J_k E \xrightarrow{\overline{\lambda}_k} VJ_k E,$$

and let us set

$$\overline{\lambda}_k = \sum_{|\alpha| \le k} a^{(\lambda, \alpha)} \partial / \partial y_\alpha^\lambda.$$

The coefficients $a^{(\lambda, \alpha)}$ are linear forms defined on the vector bundle $J_{\ell+k}^0 TP \times_P J_k E$ and it is evident that $W \in \mathbf{R}_{\ell+k}^0$ if and only if $a^{(\lambda, \alpha)}(W) = 0$ for any pair (λ, α). The local basis $\{\epsilon_i^\beta\}$ of $J_{\ell+k}^0 TP|U$ lifts, by reciprocal image, onto a local basis of the fibre bundle

$$J_{\ell+k}^0 TP \times_P \alpha_k^{-1}(U).$$

Let us now write

$$a^{(\lambda, \alpha)} = \sum a_{(\beta, i)}^{(\lambda, \alpha)} (\epsilon_i^\beta)^*,$$

where $\{(\epsilon_i^\beta)^*\}$ is the dual basis to $\{\epsilon_i^\beta\}$. The matrix of $\overline{\lambda}_k$ [1] with respect to the local bases $\{\epsilon_i^\beta\}$ of $J_{\ell+k}^0 TP \times_P \beta_k^{-1}(\mathcal{U})$ and $\{\partial / \partial y_\alpha^\lambda\}$ of $V\beta_k^{-1}(\mathcal{U}) \subset VJ_k E$ is no other than $a_{(\beta, i)}^{(\lambda, \alpha)}$, a matrix whose coefficients are functions defined on the open set $\beta_k^{-1}(\mathcal{U})$. Furthermore, the dimension of $(\mathbf{R}_{\ell+k}^0)_Z$ is equal to the co-rank of this matrix since the above isotropy is the common kernel of the forms $a^{(\lambda, \alpha)}$. Let us now take a point $W \in \beta_{k+1}^{-1}$ $(\mathcal{U}) \subset J_{k+1} E$ projecting in Z and let us indicate by

$$\left(b_{(\beta, i)}^{(\lambda, \alpha)} \right), \ 1 \le |\alpha| \le k + 1, \quad 1 \le |\beta| \le k + 1,$$

the matrix of $\overline{\lambda}_{k+1}$. The commutative diagram

---

[1] not to be confounded with the indices λ.

$$J^0_{\ell+k+1}TP \times_P J_{k+1}E \xrightarrow{\overline{\lambda}_{k+1}} VJ_{k+1}E$$

$$\downarrow \qquad\qquad\qquad \downarrow T\rho$$

$$J^0_{\ell+k}TP \times_P J_k E \xrightarrow{\overline{\lambda}_k} VJ_k E$$

shows that $T\rho \circ \overline{\lambda}_{k+1}$ factors to $J^0_{\ell+k}TP \times_P J_k E$ onto the morphism $\overline{\lambda}_k$ and, consequently, the matrix of $\overline{\lambda}_{k+1}$ transcribes by

$$\begin{pmatrix} B & C \\ A & 0 \end{pmatrix},$$

where A is the matrix of $\overline{\lambda}_k$ and the blocks $(B\ C)$ are formed by the components of the forms $b^{(\lambda,\alpha)}$, $|\alpha| = k+1$, the block C being formed by the $b^{(\lambda,\alpha)}_{(\beta,\,j)}$, with $|\beta| = \ell + k + 1$. The coordinates being fixed, we can identify $(J^0_{\ell+k}TP)_z$ to a direct factor of $(J^0_{\ell+k+1}TP)_z$, the forms $a^{(\lambda,\alpha)}_Z$ then becoming linear forms on the second space. Under these conditions, the surjectivity of $(\mathbf{R}^0_{\ell+k+1})_W \to (\mathbf{R}^0_{\ell+k})_Z$ simply means that the quotient sub-space $[b^{(\lambda,\alpha)}_W]/[a^{(\lambda,\alpha)}_Z]$ projects injectively into the space $[\epsilon^\beta_i(z)^*]_{|\beta|=k+1}$, where [ ] indicates the generated sub-space. However, the rank of this projection is given by the rank of the block C_W and, consequently, the surjectivity can be translated by the following condition: The rank, at the point W, of the above matrix is equal to the rank of

$$\begin{pmatrix} 0 & C \\ A & 0 \end{pmatrix}.$$

Since the rank of C can vary, this condition cannot be translated, locally, by the vanishing of a certain number of determinants. We remark nevertheless that $(\mathfrak{g}_{\ell+k+1})_W$ is the simultaneous kernel of the

restrictions, to the total symbol, of the forms $b^{(\lambda,\alpha)}$, $|\alpha| = k + 1$, hence the rank of C_W, equal to the co-dimension of $(\mathfrak{g}_{\ell+k+1})_W$, is constant above a trajectory $\mathbf{F}_0$ of E, where after, we can determine locally, in the reciprocal image $\beta_{k+1}^{-1}(\mathbf{F}_0)$, the equality of the ranks of these two matrices by means of the nullity of a certain number of determinants. The equation Θ_{k+1} is therefore closed in $\beta_{k+1}^{-1}(\mathbf{F}_0)$. In particular, $\Theta_{k+1}(\mathbf{F}_0)$ is closed in $J_{k+1}\mathbf{F}_0$.

The above reasoning amounts to a long elimination process. In fact, if we write explicitly the linear equations, bearing on the components ξ_β^i of a jet $j_{\ell+k+1}\xi(z) \in J_{\ell+k+1}^0 TP$, that define the sub-space $(\mathbf{R}_{\ell+k+1}^0)_W$, then the projection of this isotropy in $(\mathbf{R}_{\ell+k}^0)_Z$ will have, besides the equations defining $(\mathbf{R}_{\ell+k}^0)_Z$, all those that arise from the equations $b^{(\lambda,\alpha)} = 0$, $|\alpha| = k + 1$, by the elimination of the unknowns ξ_β^i, $|\beta| = \ell + k + 1$. However, this elimination consists in taking the linear and linearly independent relations among the lines of C namely, the linear combinations that annihilate the lines of C and rewrite these same relations with the help of the lines of B. We shall thus obtain the auxiliary equations that define the projection of $(\mathbf{R}_{\ell+k+1}^0)_W$. The equality of rank condition for these two matrices means precisely that the auxiliary equations are linear combinations of the equations defining $(\mathbf{R}_{\ell+k}^0)_Z$ or, equivalently, the equations whose coefficients are given by the lines of A. We thus find (apparently) the arguments of [15], p.297, which are performed at first order and for the almost-structures modelled on a transitive Lie pseudo-group.

Proposition 5. *When* $(\mathbf{F}_0, \pi, P, p)$ *is the prolongation space subordinate to an orbit* $\mathbf{F}_0$ *contained in E, then the equation* $\Theta_{k+1}(\mathbf{F}_0)$ *in* $J_{k+1}\mathbf{F}_0$ *is a closed sub-set locally defined by the vanishing of a certain*

number of determinants namely, those that establish the equality of ranks of the previously indicated matrices. Moreover, the rank of the first matrix is equal to the fibre co-dimension of $\mathbf{R}^0_{\ell+k+1}|J_{k+1}\mathbf{F}_0$ *and the rank of the sub-matrix C is equal to the co-dimension of* $\mathfrak{g}_{\ell+k+1}(\mathbf{F}_0)$.

A prolongation space (E, π, P, p) of order ℓ is said to be *homogeneous* when Γ_ℓ or, equivalently, $\prod_\ell P$ operates transitively on E. Examples of such spaces are the total spaces of ℓ-th order **G**-structures as well as those of almost-structures modelled on a transitive Lie pseudo-group and all the previous results transcribe in this context where we replace $\mathbf{F}_0$ by $E\ (= \mathbf{F}_0)$. Moreover, the only proper subordinate prolongation spaces are the infinitesimal orbits in E. We terminate this section by resuming all the data concernig formal transitivity.

Theorem 5. *Let* (E, π, P, p) *be a finite prolongation space of order* ℓ *and* $(\mathbf{F}_0, \pi, P, p)$ *the prolongation space subordinate to an orbit* $\mathbf{F}_0$ *contained in E. Let us denote by* $\eta_2 = \eta_2(\mathbf{F}_0)$ *the integer where after the symbol* $\mathfrak{g}_{\ell+k}(\mathbf{F}_0)$, $k \geq \eta_2$, *becomes 2-acyclic and by* η_∞ *the integer where after this symbol becomes involutive. Under these conditions*:

(i) *The non-linear equations* $\mathcal{R}_{\ell+k+h}(S)$, $h \geq 0$, *associated to every solution S of an* $\mathbf{F}_0$-*admissible fundamental equation of order* $k + 1 \geq \eta_2 + 1$, *are transitive (in* $(\alpha(S))$ *Lie sub-groupoids that further are closed, locally trivial and regularly embedded in* $\prod_{\ell+k+h} \alpha(S)$, $\mathcal{R}_{\ell+k+h+1}(S) = p\mathcal{R}_{\ell+k+h}(S)$ *and the morphism*

$$\mathcal{R}_{\ell+k+h+1}(S) \longrightarrow \mathcal{R}_{\ell+k+h}(S)$$

is a submersion. If, moreover, the isotropy groups $(\mathcal{R}^0_{\ell+k+h}S)_y$ *at a point* $y \in \alpha(S)$ *are all connected (or else project one upon the other), then the above submersions become surjective and the equation* $\mathcal{R}_{\ell+k}(S)$ *will be formally integrable. This being the case, the k-th order prolongation of the structure S is also formally transitive above the open set* $\alpha(S)$.

(ii) *The α-connected components $\mathcal{R}_{\ell+k+h}(S)_0$ also verify the same regularity properties, the transitivity taking place in each connected component of $\alpha(S)$ hence everywhere when S is connected. Finally, the prolongation-projection properties stated above also transcribe for the connected components namely, $\mathcal{R}_{\ell+k+h+1}(S)_0 = p\mathcal{R}_{\ell+k+h}(S)_0$ and*

$$\mathcal{R}_{\ell+k+h+1}(S)_0 \longrightarrow \mathcal{R}_{\ell+k+h}(S)_0$$

is a submersion or, in other terms, the equation $\mathcal{R}_{\ell+k}(S)_0$ is formally integrable and becomes involutive when $k \geq \eta_\infty$.

The proof is essentially provided by all the preceding sorites.

Let us next observe that the fibres of the fibration

$$\mathcal{R}^0_{\ell+k+h+1}(S) \longrightarrow \mathcal{R}^0_{\ell+k+h}(S)$$

are affine sub-spaces. Consequently, if for some integer h the isotropy $\mathcal{R}^0_{\ell+k+h}(S)_y$ is connected, the same will occur for all higher order isotropies at that point and the transitivity of $\mathcal{R}_{\ell+k+h}(S)$ will further imply that $\mathcal{R}^0_{\ell+k+h}(S)_{y'}$ is also connected at any other point $y' \in \alpha(S)$.

Corollary 12. *The given data and the hypotheses being those of the theorem, let us assume further that $\mathcal{R}^0_{\ell+k}(S)_y$ be connected at a point $y \in \alpha(S)$. Then $\mathcal{R}_{\ell+k}(S)$ is formally integrable and the k-th order prolongation of any solution S of $\mathbf{F}_{k+1}$ is a formally transitive structure.*

We thus see that the infinitesimal formal transitivity properties imply the same properties for the finite formal transitivity as long as we restrict the finite order equivalences to the α-connected components of the non-linear Lie equations associated to the structure S. Whenever one of the isotropy groups is connected, all the other goups are also connected and the formal transitivity is assured without any restriction.

We can equally define the $(k+1)-$st order equations $\widetilde{\Theta}_{k+1}$ by considering the set of all $Z \in J_{k+1}E$ such that the morphism

$$(\mathcal{R}^0_{\ell+k+1})_Z \longrightarrow (\mathcal{R}^0_{\ell+k})_{\rho Z}$$

is surjective. These equations are obviously invariant by the action of $\Gamma_{\ell+k+1}$ and $\prod_{\ell+k+1} P$ and $\widetilde{\Theta}_{k+1} \subset \Theta_{k+1}$. Moreover, every solution of a finite fundamental equation of order $k+1$ contained in $\widetilde{\Theta}_{k+1}$ verifies the following two properties:

(a) $\mathcal{R}_{\ell+k+1}(S) \longrightarrow \mathcal{R}_{\ell+k}(S)$ is surjective, and

(b) $\mathcal{R}_{\ell+k}(S)$ is homogeneous of order k.

The fact that S is a solution of a fundamental equation implies (Lemma 8) that both equations are "good" Lie sub-groupoids. Taking a finite trajectory $\mathbf{F}_0$ in E as well as the corresponding subordinate prolongation space, we shall obtain all the structures admitting formally transitive prolongations - at least for each connected component of such structures - as the solutions of $\mathbf{F}_0$-admissible fundamental equations (i.e., contained in $\widetilde{\Theta}_{k+1} \cap J_{k+1}\mathbf{F}_0$), in applying the non-linear version of Quillen's theorem at the level of 2-acyclicity. Finally, if we ascend to the level of involutivity, we shall obtain involutive equations and consequently transitive structures, for short, in the analytic case. Besides, it is clear that the solutions of these fundamental equations are, up to a prolongation, infinitesimally formally transitive and that the preceding results concerning the properties of $\mathbf{F}_0$-admissible fundamental equations can be entirely transcribed when the admissibility is taken with respect to $\widetilde{\Theta}_{k+1}$ since the conditions (a) and (b) for the non-linear equations imply the corresponding conditions for the linear equations. Unfortunately, The equations $\widetilde{\Theta}_{k+1}$ are not susceptible of a good analytical description since the surjectivity of the

isotropy groups is more likely a global topological problem involving connected components once this surjectivity is verified at the Lie algebra level, i.e., when the linear sequence

$$(\mathbf{R}^0_{\ell+k+1})_Z \longrightarrow (\mathbf{R}^0_{\ell+k})_{\rho_k Z} \longrightarrow 0$$

is exact.

We finally remark that all the preceding definitions and results are based on the general pseudo-group $\Gamma(P)$ of P and the groupoids $\prod_{\ell+k} P$. We can however and without any additional difficulties by simply adapting the definitions, re-write the entire section in the restricted context though it seems unavoidable, due to the nature of the various concepts, to consider no other but the transitive pseudo-groups. In fact, when Γ is a transitive Lie pseudo-group of order k_0 operating on the manifold P and $\mathcal{L}$ the corresponding infinitesimal pseudo-algebra, we can replace the general equivalence problem by the restricted one, the fundamental equations then becoming the finite or infinitesimal trajectories of the prolonged pseudo-groups or pseudo-algebras $\Gamma_{\ell+k}$ and $\mathcal{L}_{\ell+k}$. In much the same way, we shall replace, in the notions of homogeneity and transitivity, the general equivalence by the restricted one limiting ourselves to the elements of $J_{\ell+k}\Gamma$ and $J_{\ell+k}\mathcal{L}$.

3. Formally Transitive Structures

Let (E, π, P, p) be a finite prolongation space of order ℓ that we assume, from now on, homogeneous or else we restrict our attention to a subordinate prolongation space relative to an orbit of order zero.

We next consider a formally homogeneous structure S of species E and will say that another structure S' is formally equivalent to S when, for all integers $k \geq 0$ and for any pair of points $y \in \alpha(S)$ and $y' \in \alpha(S')$, there exists $Y \in \prod_{\ell+k}$ such that $Y(j_k S(y)) = j_k S'(y')$ or, in other terms, when the two jets belong to the same k-th order finite orbit namely, the orbit containing $im\ j_k S$. The jet Y will be named a k-th order finite

equivalence. Further, we shall say that the formal equivalence is *dominated* when there exists an integer ν_0 such that every finite equivalence Y of order $k \geq \nu_0$ is "dominated" by a finite equivalence $\widetilde{Y}$ of order $k+1$, where $\rho_k \widetilde{Y} = Y$. This being so, every finite equivalence of order $k \geq \nu_0$ is induced, at order k, by an infinite order equivalence namely, by an infinite jet $(Y_h)_{h \geq 0}$, where Y_h is a finite equivalence of order h and $\rho_h Y_{h+1} = Y_h$. Such an infinite jet is, in view of the Theorem of Borel, the infinite jet $j_\infty \varphi(y)$ of a local diffeomorphism φ of the manifold P. The notion of dominated formal equivalence is actually the notion of equivalence most often adopted in the study of **G**-structures. The recurrent construction of principal bundles whose elements are the k-th order equivalences of the flat model with the **G**-structure, this being achieved by requiring the nullity of the successive structure tensors or else by equivalent conditions imposed on the fundamental forms where after culminating in the desired dominated formal equivalence

We shall say that two germs of structures of species E are formally equivalent when they admit formally equivalent representatives, the dominated equivalence being defined similarly with the help of representatives. The study of the formal equivalence of formally infinitesimally transitive structures is a consequence of the Corollary 15.

Theorem (of formal equivalence) 6. *Let S be a connected infinitesimally formally transitive structure of type E and $\eta_2 = \eta_2(S)$ the integer where after the symbol $\mathfrak{g}_{\ell+k}(S)$, $k \geq \eta_2$, is 2-acyclic. In order that a structure S' be formally equivalent to S it is necessary and sufficient that S' be a solution of the fundamental equation $\mathbf{F}_{\eta_2+1}$ that contains im $j_{\eta_2+1}S$. Such a structure S' is infinitesimally formally transitive and its symbol $\mathfrak{g}_{\ell+k}(S')$ has, for all $k \geq 0$, the same homological properties of the initial one, the formal equivalence being a consequence of the finite equivalence of order $\eta_2 + 1$. Moreover, if S or one of its prolongations is formally transitive, that will notably take place when the non-linear*

isotropy groups $(\mathcal{R}^0_{\ell+k}S)_y$ *at a point* y_0 *project, for* $k \geq k_0$, *one upon the other or else when one of these groups is connected, then the same will hold for S' and the formal equivalence will be of dominated type.*

The structure S being connected, it is also homogeneous at any order and consequently *im* $j_{\eta_2+1}S$ is contained in a fundamental equation $\mathbf{F}_{\eta_2+1}$ that is admissible due to the (η_2+1)-st order infinitesimal transitivity of S. The uniformity of the homological properties of the symbols implies that $\mathfrak{g}_{\ell+k} = \mathfrak{g}_{\ell+k}(S)$ is 2-acyclic for $k \geq \eta_2$ and the Corollary 15 then shows that S and S' are, for all $k \geq 0$, solutions of $p_k\mathbf{F}_{\eta_2+1}$, which is the admissible fundamental equation, orbit of *im* $j_{\eta_2+k+1}S$. The property relative to the symbols $\mathfrak{g}_{\ell+k}(S')$ is an immediate consequence of the Lemma 5. Finally, the part concerning the formal transitivity of S' as well as the dominated equivalence result by an easy argument on the isotropy groups. We shall remark, for that matter, that the set of all the k-th order equivalences with a given source y and a given target y' is a homogeneous space of the isotropy groups $(\mathcal{R}^0_{\ell+k}S)_y$ and $(\mathcal{R}^0_{\ell+k}S')_{y'}$.

Remark. The connectedness hypothesis of the structure S can be replaced by the formal homogeneity or even by the homogeneity of order η_2+1, in such a way as to assure that *im* $j_{\eta_2+1}S$ be contained within an orbit.

Corollary 13. *Let* S *be a connected structure or a formally homogeneous structure whose* μ-*th prolongation is infinitesimally transitive and let* η_2 *be the integer from where on all the symbols* $\mathfrak{g}_{\ell+k}(S)$, $k \geq \eta_2$, *become 2-acyclic. In order that* S' *be formally equivalent to S, it is necessary and sufficient that S' be a solution of the fundamental equation* $\mathbf{F}_{k+1}$, $k = \max\{\mu, \eta_2\}$, *that contains im* $j_{k+1}S$. *The* μ-*th prolongation of such a structure S' is infinitesimally formally transitive. If, further, one of the prolongations of S is formally transitive, the same will occur with S' and the formal equivalence will be dominated.*

The formal problem being resolved, we can proceed with our inquiry and look at what really matters, namely, *the local equivalence* of the two structures. But here there is no way out other than to integrate the fundamental equations.

As for non-transitive structures, we should point out that much can be said in the case where the pseudo-group of local automorphisms or the pseudo-algebra of infinitesimal automorphisms have regular orbits and, more precisely, when these orbits are distributed within a regular foliation. We can even assume that their prolongations to *E* provide as well regular foliations. In this case we can define, at least locally, quotient prolongation spaces and quotient structures and, adapting accordingly the definitions, we shall be able to rewrite most of what was stated in this and in the previous section and consequently study the local as well as the global equivalence problems. We should also mention that Pradine's *holonomy groupoid* is certainly a powerful tool that will hopefully determine the equivalences.

We finally say a few words on a rather outstanding example of a transitive and homogeneous structure, one of the many *Chef – d'Oeuvres* given to us by Élie Cartan ([3], [4]). We are referring here to the *Systèmes de Pfaff en Drapeau* or, in celtic, *Flag Systems*.

These are very special non-integrable Pfaffian systems enjoying the property of "slowly increasing their manifestation" of non-integrability or, in other terms, the successive derived systems decrease their dimensions just by one unit and, of course, terminate by the null system. The most remarkable about these systems is the fact that they provide a vivid example of one of Cartan's *merihedric* (*mériédrique*) prolongation and equivalence processes. Since all this can already be learnt in [11], [10], and [8], we shall only tell "a nice story" about them. First of all, and this is the main fact, the automorphism groups and algebras of arbitrary Flag Systems are all canonically isomorphic and, more specifically, they are all canonically isomorphic to the *Darboux automorphisms* emanating from the equation $\mathbf{dx}^2 - \mathbf{x}^3\mathbf{dx}^1 = \mathbf{0}$. We can however study and classify these structures with the help of the isotropy groups and algebras of arbitrary

orders. These are intrinsically defined objects and the classification comes out upon looking at the differences in the *co-ranks* of these (Fréchet infinite dimensional) groups and algebras at least for those flags that the author in [9] calls *elementary*. When the systems are non-elementary, their classification becomes much more delicate and a new phenomenon does occur namely, the appearance of continuous or even differentiable deformations (variations) of a given flag via non-equivalent flags. However, we can also cope with such phenomena by considering again the isotropy algebras as discrete (finite number of variables) moduli. Most authors still employ the rather clumsy and outdated *pseudo-normal forms* introduced, way back in 1980, by the present author though the above terminology is due to others. At present, we can do no better than apologize for having considered these silly and uncouth objects and where several authors come forth with fathom deep proofs that nobody will ever read nor understand except, and with all the blessing and mercy of the All Mighty, those authors that wrote them.

And what about the totally intransitive or rigid structures. For this, we shall be forced to "admire the fragile and beautiful butterfly that opens its wings in Aix-en-Provence and causes not the display of similar wings in Fukushima but a tsunami". This being a rather long subject we shall not even try to say a word about it hoping that other authors give us much to read and enjoy.

References

[1]   U. Amaldi, Introduzione alla teoria dei gruppi continui infiniti di trasformazioni, Libreria dell'Università, Roma, 1942.

[2]   E. Cartan, Sur la structure des groupes infinis de transformation, Annales Sci. École Norm. Sup. 21 (1904), 153-206.

[3]   E. Cartan, Les systèmes de Pfaff à cinq variables et les équations aux derives partielles du second ordre, Annales Sci. École Norm. Sup. 27(1910), 109-192.

[4]   E. Cartan, Sur l'équivalence absolue de certains systèmes d'équations différentielles et sur certaines familles de courbes, Bull. Soc. Math. de France 42 (1914), 12-48.

[5]   A. Kumpera, Étude des invariants différentiels attachés à un pseudogroupe de Lie, Thèse, Paris, 1967.

[6] A. Kumpera, Invariants différentiels d'un pseudogroupe de Lie, I, II. J. Differ. Geom. 10(279-335) (1975), 347-417.

[7] A. Kumpera, On the Lie and Cartan theory of invariant differential systems, J. Math. Sci. Univ. Tokyo 6 (1999), 229-314.

[8] A. Kumpera, Automorphisms of Flag Systems, arXiv, 2014.

[9] A. Kumpera, On the equivalence problem for geometric structures, I. pre-print, 2014.

[10] A. Kumpera and J. Rubin, Multi-flag systems and ordinary differential equations, Nagoya Math. J. 166 (2002), 1-27.

[11] A. Kumpera and C. Ruiz, Sur l'équivalence locale des systèmes de Pfaff en drapeau. F. Gherardelli (Ed.), Monge-Ampère Equations and Related Topics, Firenze 1980, Proceedings, Istit. Naz. Alta Mat. Francesco Severi, Roma, (1982), 201-248.

[12] A. Kumpera and D. Spencer, Lie Equations, Vol. 1: General Theory, Princeton University Press, 1972.

[13] P. Molino, Sur quelques propriétés des *G*-structures, J. Differ. Geom. 6 (1972), 489-518.

[14] J. Pommaret, Théorie des déformations de structures, Ann. Inst. Henri Poincaré 18 (1973), 285-352.

[15] E. Vessiot, Théorie des groupes continues, Ann. Sci. École Norm. Sup. 20 (1903), 411-451.

∎